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THE BOUNDARY ELEMENT METHOD FOR THICK PLATES ON A WINKLER FOUNDATION

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ABSTRACT

In this paper the application of the boundary element method to thick plates resting on a Winkler foundation is presented. The Reissner plate bending theory is used to model the plate behaviour. The Winkler foundation model is represented by continuous springs which are directly incorporated into the governing differential equation. The fundamental solutions are constructed using operator decoupling technique. These fundamental solutions represent three different cases depending on the problem constants. The explicit forms of the boundary and internal point kernels are given in all cases. Quadratic isoparametric boundary elements are used to model the plate boundary. Several examples are presented to demonstrate the accuracy of the present formulation. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: Reissner's plate theory; Winkler foundation; boundary element method; fundamental solutions

1. INTRODUCTION

The theory of thick plates on a Winkler foundation has many applications in structural analysis of foundation plates. The usual way of analysing such a model is to represent the foundation plate using the thin plate bending theory.[1] The soil is usually represented by a set of discrete spring elements. However, a continuous subgrade reaction is desirable for more accurate analysis.

The early applications of the BEM to thin plates on an elastic foundation is by Katsikadelis and Armenakas[2] and Balaš and Sládek.[3] In the thin plate theory, the problem is represented with only two degrees of freedom (the deflection and the normal slope) which results in a smaller computer storage requirement than that of the thick plate theory which has three degrees of freedom. However, in the thin plate theory, additional unknowns are needed due to the Kirchhoff shear at corners. Costa and Brebbia[4] considered thin plates on a Winkler foundation. They expressed the corner forces in terms of the normal slope of the adjacent nodes to avoid the introduction of additional corner unknowns. Puttonen and Varpasuo[5] expressed the fundamental solution for the thin plate on an elastic foundation in terms of Fourier–Bessel integrals, which leads to a more complicated and unexplicit forms of the fundamental solution kernels. The corner term was ignored in the formulation presented in Reference 5.

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Bezine[6] employed the fundamental solution for thin plates (the bi-harmonic equation) in the analysis of thin plates on an elastic foundation. An additional domain integral is required in that formulation, which is undesirable in the BEM. Kamiya and Sawaki[7] applied the dual reciprocity approximation in order to avoid domain integration in the Bezine formulation. However, this approach requires additional internal nodes in the domain, particularly in the case of a totally clamped plate.

In the case of foundation plates, usually the plate has a considerable thickness. For more accurate results, a refined plate model could be used. The Reissner plate bending theory (thick plate theory)[8] can be considered as an alternative for such applications. The advantages of thick plate theory are the inclusion of the transverse shear deformation and that it expresses the problem in terms of three physical boundary conditions avoiding the corner force approximation. Furthermore, it has been demonstrated[9] that thick plate theory can be used to analyse both thin and thick plates.

Frederick[10] extended Reissner's formulation to plates on a Winkler foundation. A series solutions for a rectangular simply supported plates on a Winkler foundation was also developed in Reference 10.

Recently, El-Zafrany *et al.*[11] developed a BEM for thin plates on a Winkler foundation. In their formulation, three degrees of freedom per node was considered, which leads to the same storage requirements as that for thick plate problems. They showed that using three degrees of freedom per node leads to more accurate results. This makes the use of the Reissner plate model for plates on a Winkler foundation more attractive.

Wang *et al.*[12] extended the work of Balaš and Sládek[3] to deal with thick plates resting on a Winkler foundation. They showed that the fundamental solution has three different cases depending on the parameters of the plate and the foundation. However, their work is incomplete as only one of the three possible cases for the fundamental solutions was considered. Also, cell discretization was used[12] to compute the domain integral due to body force.

In this paper, the other two cases of the fundamental solution are derived for the Reissner plate resting on a Winkler foundation. The singularity behaviour for the fundamental solutions (including the case considered by Wang *et al.*[12]) are discussed for the first time. A semi-analytical procedure based on a Taylor series expansion is proposed to deal with the strong singularity $O(1/r)$. The explicit forms of the internal point kernels are derived for all cases of the fundamental solutions. In this work the body force domain integrals are represented by equivalent boundary integrals and not by cell discretization as in Reference 12. Quadratic isoparametric elements are used. Several examples, including free edge boundary conditions, are presented to demonstrate the accuracy and the validity of the present model.

2. BASIC EQUATIONS

In this section, the basic equations for a Reissner plate resting on a Winkler foundation are reviewed. Throughout this paper, the indicial notation is used. Comma denotes differentiation $((\cdots)_{,\theta} = \partial_\theta(\cdots) = \partial(\cdots)/\partial x_\theta)$ and $(\cdots)_{,n}$ denotes the derivative with respect to the normal n. Greek indices will vary from 1 to 2, whereas Roman indices vary from 1 to 3.

Consider an arbitrary plate of thickness h in the x_i space. The x_1–x_2 plane is assumed to be located at the middle surface $x_3 = 0$. The generalized displacements are denoted as u_i, where, u_α denotes rotations (ϕ_{x_1} and ϕ_{x_2}) and u_3 the transverse deflection w in the x_3 direction. According to Reissner,[8] the stress–displacement relationships are given by

$$\begin{aligned} M_{\alpha\beta} &= D\frac{1-v}{2}\left(u_{\alpha,\beta}+u_{\beta,\alpha}+\frac{2v}{1-v}u_{\gamma,\gamma}\delta_{\alpha\beta}\right) \\ Q_{\alpha} &= D\frac{1-v}{2}\lambda^2(u_\alpha+u_{3,\alpha}) \end{aligned} \tag{1}$$

where $M_{\alpha\beta}$ and Q_α are the bending and shear stresses, respectively, and $D=Eh^3/12(1-v^2)$ is the plate flexural rigidity, E is the Young's modulus, v is the Poisson's ratio and $\lambda=\sqrt{10}/h$ is the shear factor. It has to be noted that the effect of the transverse normal stresses on the bending moment is ignored[12] in equation (1). In Section 10, it is demonstrated that this term has no effect on the accuracy of the results.

The equilibrium equations can be written as follows:

$$\begin{aligned} M_{\alpha\beta,\beta}-Q_\alpha=0 \\ Q_{\alpha,\alpha}+q-k_\mathrm{f}u_3=0 \end{aligned} \tag{2}$$

where q is the distributed load per unit area, $k_\mathrm{f}u_3$ s the interface pressure between the plate and the foundation[13] and k_f is the modulus of subgrade reaction. The generalized tractions at boundary point can be defined as

$$\begin{aligned} p_\alpha &= M_{\alpha\beta}n_\beta \\ p_3 &= Q_\alpha n_\alpha \end{aligned} \tag{3}$$

where the n_β are the components of the outward normal vector to the plate boundary Γ.

The generalized Navier equations can be formed by substituting (1) into (2) to give

$$L_{ij}u_j+b_i=0 \tag{4}$$

where

$$\begin{aligned} L_{\alpha\beta} &= \left(D\frac{(1-v)}{2}\nabla^2-C\right)\delta_{\alpha\beta}+D\frac{(1+v)}{2}\partial_\alpha\partial_\beta \\ L_{\alpha 3} &= -C\partial_\alpha \\ L_{3\beta} &= C\partial_\beta \\ L_{33} &= C\nabla^2-k_\mathrm{f} \end{aligned} \tag{5}$$

and

$$C=D\left(\frac{1-v}{2}\right)\lambda^2$$

in which L_{ij} is the generalized Navier differential operator, the $b_i=q\delta_{i3}$ are the generalized body force components, $\delta_{\alpha\beta}$ is the Kronecker delta function, and ∇^2 is the two-dimensional Laplace operator.

3. GENERALIZED SOMIGLIANA'S IDENTITY

A suitable weighted residual statement[14] for equation (4) can be written as follows:

$$\int_\Omega (L_{ij}u_j+b_i)U_i\,\mathrm{d}\Omega=0 \tag{6}$$

where U_i are the weighting functions.

Integrating by parts a suitable number of times and making use of the stress–displacement relationships (1), the following expression can be written:

$$\int_\Gamma P_j u_j \,\mathrm{d}\Gamma = \int_\Gamma U_j p_j \,\mathrm{d}\Gamma + \int_\Omega [(L^{\mathrm{adj}}_{jk} U_k) u_j + U_j b_j] \,\mathrm{d}\Omega \tag{7}$$

where L^{adj}_{jk} is the adjoint operator of L_{jk} and is given by

$$\begin{aligned} L^{\mathrm{adj}}_{\alpha\beta} &= \left(D\frac{(1-\nu)}{2}\nabla^2 - C \right)\delta_{\alpha\beta} + D\frac{(1+\nu)}{2}\partial_\alpha\partial_\beta \\ L^{\mathrm{adj}}_{\alpha 3} &= C\partial_\alpha \\ L^{\mathrm{adj}}_{3\beta} &= -C\partial_\beta \\ L^{\mathrm{adj}}_{33} &= C\nabla^2 - k_{\mathrm{f}} \end{aligned} \tag{8}$$

Assuming the (U, P) state represents the fundamental state (unit generalized load at the source point $\mathbf{X}' \in \Omega$ in an arbitrary direction i, i.e.

$$L^{\mathrm{adj}}_{jk} U_{ik}(\mathbf{X}', \mathbf{X}) = -\delta(\mathbf{X}', \mathbf{X})\delta_{ij} \tag{9}$$

where $\delta(\mathbf{X}', \mathbf{X})$ is the Dirac delta function, $\mathbf{X} \in \Omega$ is the field point, and $U_{ik}(\mathbf{X}', \mathbf{X})$ are the two-point fundamental solutions which represent the displacement at the point $\mathbf{X}$ in the direction k due to the unit generalized impulse acting at an internal source point $\mathbf{X}'$ in an arbitrary direction i in an infinite plate resting on a Winkler foundation.

After making use of the properties of the Dirac delta function, equation (7) can be rewritten as

$$u_j(\mathbf{X}') + \int_\Gamma P_{ij}(\mathbf{X}', \mathbf{x}) u_j(\mathbf{x}) \,\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma U_{ij}(\mathbf{X}', \mathbf{x}) p_j(\mathbf{x}) \,\mathrm{d}\Gamma(\mathbf{x}) + \int_\Omega U_{ij}(\mathbf{X}', \mathbf{X}) b_j(\mathbf{X}) \,\mathrm{d}\Omega(\mathbf{X}) \tag{10}$$

where $\mathbf{x} \in \Gamma$ is an arbitrary field point on the boundary. Equation (10) represents the generalized Somigliana identity for a Reissner plate resting on the Winkler foundation.

4. FUNDAMENTAL SOLUTIONS

In order to obtain the fundamental solution for the operator in equation (9), an operator decoupling process is carried out. Following Hörmander,[15] equation (9) can be written in the following form:

$$U_{ij}(\mathbf{X}', \mathbf{X}) = {}^{\mathrm{co}}L^{\mathrm{adj}}_{ji} \phi(\mathbf{X}', \mathbf{X}) \tag{11}$$

where $\phi(\mathbf{X}',\mathbf{X})$ is an unknown scalar potential and ${}^{\text{co}}L_{ij}^{\text{adj}}$ is the co-factor matrix of the operator L_{ij}^{adj} and given by

$$\begin{aligned}
{}^{\text{co}}L_{\alpha\beta}^{\text{adj}} &= DC\left[\nabla^4 - \frac{k_{\text{f}}}{C}\nabla^2 + \frac{k_{\text{f}}}{D}\right]\delta_{\alpha\beta} - \partial_\alpha\partial_\beta\left[DC\frac{1+v}{2}\nabla^2 + C^2 - Dk_{\text{f}}\frac{1+v}{2}\right] \\
{}^{\text{co}}L_{\alpha 3}^{\text{adj}} &= C\partial_\alpha\left(D\frac{1-v}{2}\nabla^2 - C\right) \\
{}^{\text{co}}L_{3\beta}^{\text{adj}} &= -C\partial_\beta\left(D\frac{1-v}{2}\nabla^2 - C\right) \\
{}^{\text{co}}L_{33}^{\text{adj}} &= (D\nabla^2 - C)\left(D\frac{1-v}{2}\nabla^2 - C\right)
\end{aligned} \tag{12}$$

Now the potential $\phi(\mathbf{X}',\mathbf{X})$ can be evaluated as follows:

$$\det[{}^{\text{co}}L_{ij}^{\text{adj}}]\phi(\mathbf{X}',\mathbf{X}) = -\delta(\mathbf{X}',\mathbf{X}) \tag{13}$$

or can be rewritten as follows:

$$\frac{\mathscr{F}}{4}\left(\nabla^4 - \frac{k_{\text{f}}}{C}\nabla^2 + \frac{k_{\text{f}}}{D}\right)(\nabla^2 - \lambda^2)\phi(\mathbf{x}',\mathbf{x}) = -\delta(\mathbf{x}',\mathbf{x}) \tag{14}$$

where

$$\mathscr{F} = D^3(1-v)^2\lambda^2 \tag{15}$$

The scalar potential $\phi(\mathbf{X}',\mathbf{X})$ can be expressed as a linear combination of two fundamental solutions:[12] $\mathscr{A}(\mathbf{X}',\mathbf{X})$ for the first operator $\frac{1}{4}\mathscr{F}(\nabla^4 - (k_{\text{f}}/C)\nabla^2 + k_{\text{f}}/D)$ and $\mathscr{B}(\mathbf{X}',\mathbf{X})$ for the second operator $\frac{1}{4}\mathscr{F}(\nabla^2 - \lambda^2)$ as follows:

$$\phi(\mathbf{X}',\mathbf{X}) = \frac{((k_{\text{f}}/C) - \lambda^2 - \nabla^2)\mathscr{A}(\mathbf{X}',\mathbf{X}) + \mathscr{B}(\mathbf{X}',\mathbf{X})}{\lambda^4 - (k_{\text{f}}/C)\lambda^2 + k_{\text{f}}/D} \tag{16}$$

If the following constants are defined:[3] $\ell^4 = D/k_{\text{f}}$ denotes the characteristic length and the dimensionless parameter $\kappa = k_{\text{f}}\ell^2/2C$. The expression for the fundamental solution $\mathscr{A}$ can be written as follows:[3]

For $\kappa > 1$ (*Case* 1):

$$\mathscr{A}(\mathbf{X}',\mathbf{X}) = \frac{-\ell^2}{\pi\mathscr{F}S}[K_0(er) - K_0(dr)] \quad \text{and} \quad S = \sqrt{\kappa^2 - 1} \tag{17}$$

where

$$d^2 = \frac{\kappa + \sqrt{\kappa^2 - 1}}{\ell^2} \quad \text{and} \quad e^2 = \frac{\kappa - \sqrt{\kappa^2 - 1}}{\ell^2}$$

For $\kappa = 1$ (*Case* 2):

$$\mathscr{A}(\mathbf{X}',\mathbf{X}) = \frac{-\ell}{\pi\mathscr{F}}rK_1\left(\frac{r}{\ell}\right) \tag{18}$$

For $\kappa < 1$ (*Case* 3):

$$\mathscr{A}(\mathbf{X}', \mathbf{X}) = \frac{\ell^2}{\mathscr{F} \sin 2\psi} \mathrm{Re}[H_0^{(1)}(\zeta r)] \tag{19}$$

where

$$\zeta = \frac{\exp(\mathrm{i}(\pi/2 + \psi))}{\ell} \quad \text{and} \quad \mathrm{i} = \sqrt{-1}$$

in which $\kappa = \cos 2\psi$, $\sqrt{1-\kappa^2} = \sin 2\psi$ and $\psi \in [0, \pi/4]$. The terms $K_0(\cdot)$, $K_1(\cdot)$, and $H_0^{(1)}(\cdot)$ are modified Bessel functions and Hankel function, respectively.[16] Physically, $\kappa > 1$ may be regarded as representing stiff soil or plates with small thickness.

On the other hand, the fundamental solution $\mathscr{B}$ is given by

$$\mathscr{B}(\mathbf{X}', \mathbf{X}) = \frac{2}{\pi \mathscr{F}} K_0(z) \tag{20}$$

where $z = \lambda r$.

The potential ϕ is obtained by substituting $\mathscr{A}$ and $\mathscr{B}$ into equation (16) and subsequently the fundamental solution is evaluated from equation (11) to give

$$U_{ij} = U_{ij}^{\mathscr{A}} + U_{ij}^{\mathscr{B}} \tag{21}$$

where the expressions $U_{ij}^{\mathscr{B}}$ do not depend on κ so that they are the same for the three fundamental solution cases and given by

$$\begin{aligned} U_{\alpha\beta}^{\mathscr{B}} &= \frac{1}{\pi D(1-\nu)}[B_1(z)\delta_{\alpha\beta} - A_1(z) r_{,\alpha} r_{,\beta}] \\ U_{\alpha 3}^{\mathscr{B}} &= 0 \\ U_{3\alpha}^{\mathscr{B}} &= 0 \\ U_{33}^{\mathscr{B}} &= 0 \end{aligned} \tag{22}$$

where

$$A_1(\cdot) = K_0(\cdot) + \frac{2}{(\cdot)} K_1(\cdot) \quad \text{and} \quad B_1(\cdot) = K_0(\cdot) + \frac{1}{(\cdot)} K_1(\cdot) \tag{23}$$

and $U_{ij}^{\mathscr{A}}$ are dependent on κ, so that it has three different expressions. It has to be noted that Wang *et al.*[12] derived the fundamental solution for Case 3 ($\kappa < 1$) only. Herein, the other two cases will be considered. For the sake of completeness the kernels for Case 3 are also listed. The expressions for $U_{ij}^{\mathscr{A}}$ are given by the following.

Case 1 ($\kappa > 1$):

$$
\begin{aligned}
U^{\mathscr{A}}_{\alpha\beta} &= \frac{\ell^2}{2\pi D(1-v)\lambda^2 S}\left\{\frac{1}{r}[\Upsilon_3 - \alpha\Upsilon_1]\delta_{\alpha\beta} - \left[(\Upsilon_4 - \alpha\Upsilon_2) + \frac{2}{r}(\Upsilon_3 - \alpha\Upsilon_1)\right] r_{,\alpha} r_{,\beta}\right\} \\
U^{\mathscr{A}}_{\alpha 3} &= \frac{-\ell^2}{4\pi DS} r_{,\alpha}\Upsilon_1 \\
U^{\mathscr{A}}_{3\alpha} &= -U^{*}_{\alpha 3} \\
U^{\mathscr{A}}_{33} &= \frac{\ell^2}{4\pi DS}\left[\frac{-2}{(1-v)\lambda^2}\Upsilon_2 + \Upsilon_0\right]
\end{aligned}
\tag{24}
$$

where

$$
\Upsilon_i = \begin{cases} e^i K_0(er) - d^i K_0(dr), & i\text{: even number} \\ e^i K_1(er) - d^i K_1(dr), & i\text{: odd number} \end{cases}
\tag{25}
$$

and $\alpha = (k_{\mathrm{f}}/C) - \frac{1}{2}(1-v)\lambda^2$.

Case 2 ($\kappa = 1$):

$$
\begin{aligned}
U^{\mathscr{A}}_{\alpha\beta} &= \frac{1}{\pi D(1-v)}\left\{\frac{1}{\lambda^2\ell^2}\left[A_1\left(\frac{r}{\ell}\right) r_{,\alpha} r_{,\beta} - \left(\frac{\ell}{r}\right) K_1\left(\frac{r}{\ell}\right)\delta_{\alpha\beta}\right]\right. \\
&\quad \left. + \frac{\overline{\alpha}}{4}\left[K_0\left(\frac{r}{\ell}\right)\delta_{\alpha\beta} - \left(\frac{r}{\ell}\right) K_1\left(\frac{r}{\ell}\right) r_{,\alpha} r_{,\beta}\right]\right\} \\
U^{\mathscr{A}}_{\alpha 3} &= \frac{-1}{4\pi D} r K_0\left(\frac{r}{\ell}\right) r_{,\alpha} \\
U^{\mathscr{A}}_{3\alpha} &= -U^{*}_{\alpha 3} \\
U^{\mathscr{A}}_{33} &= \frac{1}{4\pi D(1-v)\lambda^2}\left[4K_0\left(\frac{r}{\ell}\right) + \ell^2\lambda^2\overline{\alpha}\left(\frac{r}{\ell}\right) K_1\left(\frac{r}{\ell}\right)\right]
\end{aligned}
\tag{26}
$$

where $\overline{\alpha} = (1-v) - 2/\ell^2\lambda^2$.

Case 3 ($\kappa < 1$):

$$
\begin{aligned}
U^{\mathscr{A}}_{\alpha\beta} &= \frac{\ell^2}{4D(1-v)\lambda^2 \sin 2\psi}\left\{\frac{1}{r}[\Psi_3 + \alpha\Psi_1]\delta_{\alpha\beta} + \left[(\Psi_4 + \alpha\Psi_2) - \frac{2}{r}(\Psi_3 + \alpha\Psi_1)\right] r_{,\alpha} r_{,\beta}\right\} \\
U^{\mathscr{A}}_{\alpha 3} &= \frac{\ell^2}{8D\sin 2\psi} r_{,\alpha}\Psi_1 \\
U^{\mathscr{A}}_{3\alpha} &= -U^{*}_{\alpha 3} \\
U^{\mathscr{A}}_{33} &= \frac{-\ell^2}{8D\sin 2\psi}\left[\frac{2}{(1-v)\lambda^2}\Psi_2 + \Psi_0\right]
\end{aligned}
\tag{27}
$$

where

$$
\Psi_i = \begin{cases} 2\mathrm{Re}[\zeta^i H_0^{(1)}(\zeta r)], & i\text{: even number} \\ 2\mathrm{Re}[\zeta^i H_1^{(1)}(\zeta r)], & i\text{: odd number} \end{cases}
\tag{28}
$$

The traction fundamental solution kernels can be computed by substituting the displacement fundamental solution into equations (1) and (3), to give

$$
P_{ij} = P^{\mathscr{A}}_{ij} + P^{\mathscr{B}}_{ij}
\tag{29}
$$

where

$$
\begin{aligned}
P_{i\alpha}^{\mathscr{A},\mathscr{B}} &= \frac{D(1-v)}{2}\left(U_{i\alpha,\beta}^{\mathscr{A},\mathscr{B}} + U_{i\beta,\alpha}^{\mathscr{A},\mathscr{B}} + \frac{2v}{1-v}U_{i\theta,\theta}^{\mathscr{A},\mathscr{B}}\delta_{\alpha\beta}\right)n_\beta \\
P_{i3}^{\mathscr{A},\mathscr{B}} &= \frac{D(1-v)}{2}\lambda^2\left(U_{i\beta}^{\mathscr{A},\mathscr{B}} + U_{i3,\beta}^{\mathscr{A},\mathscr{B}}\right)n_\beta
\end{aligned}
\tag{30}
$$

It has to be noted that the differentiation should be carried out with respect to the co-ordinate of the field point $\mathbf{X}$.

Similar to the displacement fundamental solutions, the expressions $P_{ij}^{\mathscr{B}}$ do not depend on κ, and are given by

$$
\begin{aligned}
P_{\gamma\alpha}^{\mathscr{B}} &= \frac{1}{2\pi r}[zK_1(z)(2r_{,\gamma}r_{,\alpha}r_{,n} - \delta_{\gamma\alpha}r_{,n} - r_{,\alpha}n_\gamma) \\
&\quad +2A_1(z)(4r_{,\gamma}r_{,\alpha}r_{,n} - \delta_{\gamma\alpha}r_{,n} - r_{,\alpha}n_\gamma - r_{,\gamma}n_\alpha)] \\
P_{\gamma3}^{\mathscr{B}} &= \frac{\lambda^2}{2\pi}[B_1(z)n_\gamma - A_1(z)r_{,\gamma}r_{,n}] \\
P_{3\alpha}^{\mathscr{B}} &= 0 \\
P_{33}^{\mathscr{B}} &= 0
\end{aligned}
\tag{31}
$$

On the other hand the expressions $P_{ij}^{\mathscr{A}}$ depend on κ and are given by the following.

Case 1 ($\kappa > 1$):

$$
\begin{aligned}
P_{\gamma\alpha}^{\mathscr{A}} &= \frac{\ell^2}{4\pi\lambda^2 S}\left\{(\Upsilon_5 - \alpha\Upsilon_3)\left[\frac{2v}{1-v}n_\alpha r_{,\gamma} + 2r_{,\alpha}r_{,\gamma}r_{,n}\right]\right. \\
&\quad \left. + \left[\frac{4}{r^2}(\Upsilon_3 - \alpha\Upsilon_1) + \frac{2}{r}(\Upsilon_4 - \alpha\Upsilon_2)\right](4r_{,\gamma}r_{,\alpha}r_{,n} - \delta_{\gamma\alpha}r_{,n} - r_{,\alpha}n_\gamma - r_{,\gamma}n_\alpha)\right\} \\
P_{\gamma3}^{\mathscr{A}} &= \frac{\ell^2}{4\pi S}\left\{\frac{1}{r}\left(\Upsilon_3 - \frac{k_\mathrm{f}}{C}\Upsilon_1\right)(n_\gamma - 2r_{,\gamma}r_{,n}) - \left(\Upsilon_4 - \frac{k_\mathrm{f}}{C}\Upsilon_2\right)r_{,\gamma}r_{,n}\right\} \\
P_{3\alpha}^{\mathscr{A}} &= \frac{-\ell^2}{4\pi S}\left\{\Upsilon_2[(1-v)r_{,\alpha}r_{,n} + vn_\alpha] - \frac{1-v}{r}\Upsilon_1(n_\alpha - 2r_{,\alpha}r_{,n})\right\} \\
P_{33}^{\mathscr{A}} &= \frac{\ell^2}{4\pi S}\Upsilon_3 r_{,n}
\end{aligned}
\tag{32}
$$

where $r_{,n} = r_{,\alpha}n_\alpha$.

Case 2 ($\kappa = 1$):

$$P^{\mathscr{A}}_{\gamma\alpha} = \left[\frac{\overline{\alpha}}{4\pi\ell}K_1\left(\frac{r}{\ell}\right) - \frac{1}{\pi\lambda^2\ell^2 r}A_1\left(\frac{r}{\ell}\right)\right](4r_{,\gamma}r_{,\alpha}r_{,n} - \delta_{\gamma\alpha}r_{,n} - r_{,\alpha}n_\gamma - r_{,\gamma}n_\alpha)$$
$$+\left[\frac{\overline{\alpha}}{4\pi\ell}\left(\frac{r}{\ell}\right)K_0\left(\frac{r}{\ell}\right) - \frac{1-v}{2\pi\ell}K_1\left(\frac{r}{\ell}\right)\right]\left(\frac{v}{1-v}n_\alpha r_{,\gamma} + r_{,\alpha}r_{,\gamma}r_{,n}\right)$$
$$P^{\mathscr{A}}_{\gamma 3} = \frac{-1}{4\pi\ell^2}\left\{A_1\left(\frac{r}{\ell}\right)(n_\gamma - 2r_{,\gamma}r_{,n}) - \left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right)r_{,\gamma}r_{,n}\right\} \tag{33}$$
$$P^{\mathscr{A}}_{3\alpha} = \frac{1}{4\pi}\left\{(1+v)K_0\left(\frac{r}{\ell}\right)n_\alpha - \left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right)(vn_\alpha + (1-v)r_{,\alpha}r_{,n})\right\}$$
$$P^{\mathscr{A}}_{33} = \frac{r_{,n}}{4\pi\ell}\left[\left(\frac{r}{\ell}\right)K_0\left(\frac{r}{\ell}\right) - 2K_1\left(\frac{r}{\ell}\right)\right]$$

Case 3 ($\kappa < 1$):

$$P^{\mathscr{A}}_{\gamma\alpha} = \frac{\ell^2}{8\lambda^2\sin 2\psi}\left\{-(\Psi_5 + \alpha\Psi_3)\left[\frac{2v}{1-v}n_\alpha r_{,\gamma} + 2r_{,\alpha}r_{,\gamma}r_{,n}\right]\right.$$
$$\left.+\left[\frac{4}{r^2}(\Psi_3 + \alpha\Psi_1) - \frac{2}{r}(\Psi_4 + \alpha\Psi_2)\right](4r_{,\gamma}r_{,\alpha}r_{,n} - \delta_{\gamma\alpha}r_{,n} - r_{,\alpha}n_\gamma - r_{,\gamma}n_\alpha)\right\}$$
$$P^{\mathscr{A}}_{\gamma 3} = \frac{\ell^2}{8\sin 2\psi}\left\{\frac{1}{r}\left(\Psi_3 + \frac{k_\mathrm{f}}{C}\Psi_1\right)(n_\gamma - 2r_{,\gamma}r_{,n}) + \left(\Psi_4 + \frac{k_\mathrm{f}}{C}\Psi_2\right)r_{,\gamma}r_{,n}\right\} \tag{34}$$
$$P^{\mathscr{A}}_{3\alpha} = \frac{-\ell^2}{8\sin 2\psi}\left\{\Psi_2[(1-v)r_{,\alpha}r_{,n} + vn_\alpha] + \frac{1-v}{r}\Psi_1(n_\alpha - 2r_{,\alpha}r_{,n})\right\}$$
$$P^{\mathscr{A}}_{33} = \frac{\ell^2}{8\sin 2\psi}\Psi_3 r_{,n}$$

5. SINGULARITIES

By expanding the modified Bessel and Hankel functions for small argument, it can be shown that the kernels U_{ij} are weakly singular ($O(\ln r)$), whereas the kernels P_{ij} have strongly singular terms of ($O(1/r)$), i.e.

$$P_{\gamma\alpha} \to \frac{1}{2\pi r}\left\{-\delta_{\gamma\alpha}r_{,n} + \frac{1-v}{2}(r_{,\gamma}n_\alpha - r_{,\alpha}n_\gamma) + \frac{1+v}{2}(\delta_{\gamma\alpha}r_{,n} - 2r_{,\alpha}r_{,\gamma}r_{,n})\right\}$$
$$P_{33} \to \frac{-1}{2\pi r}r_{,n} \tag{35}$$

It has to be noted that the above strongly singular terms appear in each of the three cases of the fundamental solution.

6. BOUNDARY INTEGRAL EQUATION

Now equation (10) will be considered: as the internal point $\mathbf{X}'$ approaches the boundary at $\mathbf{x}'$, the following integral equation is obtained:

$$c_{ij}(\mathbf{x}')u_j(\mathbf{x}') + \fint_\Gamma P_{ij}(\mathbf{x}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma U_{ij}(\mathbf{x}',\mathbf{x})p_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) + \int_\Omega U_{ij}(\mathbf{x}',\mathbf{X})b_j(\mathbf{X})\,\mathrm{d}\Omega(\mathbf{X}) \tag{36}$$

where $\fint$ denotes a Cauchy principal value integral, and c_{ij} is the jump term that appears from the singular terms of $O(1/r)$ in the kernel P_{ij} (see the previous section).

In order to evaluate the c_{ij} terms a semi-circular region is constructed around an arbitrary collocation point $\mathbf{x}'$ which located at an arbitrary corner as shown in Figure 1. The term c_{ij} can be defined as follows:

$$c_{ij} = \lim_{\varepsilon\to 0}\int_{\Gamma^*_\varepsilon} P^*_{ij}\,\mathrm{d}\Gamma \tag{37}$$

$$r = \varepsilon, \quad r_{,n} = -1, \quad \mathrm{d}\Gamma = \varepsilon\,\mathrm{d}\varphi,$$

$$r_{,1} = -n_1 = \cos\varphi, \qquad r_{,2} = -n_2 = \sin\varphi$$

and

$$\int_{\Gamma^*_\varepsilon} \cdots\,\mathrm{d}\Gamma = \int_{\varphi_1}^{\varphi_2} \cdots \varepsilon\,\mathrm{d}\varphi \tag{38}$$

Using the above relationships and taking the limiting form of the integral in equation (37) as $\varepsilon \to 0$, the following jump term can be obtained:

$$c_{ij} = \frac{1}{2\pi}\begin{bmatrix} \Delta\varphi + \left(\frac{1+\nu}{4}\right)(\sin 2\varphi_2 - \sin 2\varphi_1) & -\left(\frac{1+\nu}{4}\right)(\cos 2\varphi_2 - \cos 2\varphi_1) & 0 \\ -\left(\frac{1+\nu}{4}\right)(\cos 2\varphi_2 - \cos 2\varphi_1) & \Delta\varphi - \left(\frac{1+\nu}{4}\right)(\sin 2\varphi_2 - \sin 2\varphi_1) & 0 \\ 0 & 0 & \Delta\varphi \end{bmatrix} \tag{39}$$

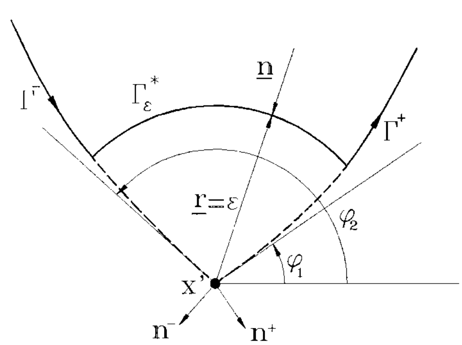

Figure 1. Corner point

where $\Delta\varphi = \varphi_2 - \varphi_1$. It has to be noted that the same result can be found in Reference 12 for Case 3. This is due to the three different cases having the same singular term. For a smooth boundary, $\varphi_1 = 0$ and $\varphi_2 = \pi$, so $c_{ij} = \frac{1}{2}\delta_{ij}$.

7. BODY FORCE INTEGRAL

The domain integral in equation (36) (due to the body force) can be transformed to the boundary for the case of uniform distributed load[17] to give

$$\int_\Omega U_{i3}(\mathbf{x}', \mathbf{X}) q \, \mathrm{d}\Omega(\mathbf{X}) = q \int_\Gamma [V_{i,n}(\mathbf{x}', \mathbf{x}) - S_{i,n}(\mathbf{x}', \mathbf{x})] \, \mathrm{d}\Gamma(\mathbf{x}) + \frac{\ell^4 q}{D} \frac{\Delta\varphi}{2\pi} \delta_{i3} \tag{40}$$

where the kernel $S_{i,n}$ is given by[17]

$$S_{i,n} = \frac{\ell^4}{2\pi D r^2} (n_i - 2r_{,i} r_{,n})(1 - \delta_{i3}) \quad \text{(no summation on } i) \tag{41}$$

and the expressions for $V_{i,n}$ are given in Appendix I.

8. NUMERICAL IMPLEMENTATION

In the present work, quadratic isoparametric elements are used to discretize the boundary of the plate. The local position of the element nodes are general, to allow for the use of continuous or discontinuous elements.

After the discretization, equation (36) can be written in a matrix form as

$$[H]\{u\} = [G]\{p\} + \{Q\} \tag{42}$$

where $[H]$ and $[G]$ are the boundary element influence matrices, $\{u\}$ and $\{p\}$ are the vectors of the boundary displacements and tractions, respectively, and $\{Q\}$ is the domain load vector. This system of algebraic equations can be solved for the boundary unknowns.

The weak singularity in the fundamental solution of the $[G]$ matrix is cancelled by using non-linear co-ordinate transformation.[18] On the other hand, the strong singularity in the $[H]$ matrix is computed in a Cauchy principal value sense via a semi-analytical procedure using a Taylor series expansion.

Consider the following integral:

$$⨍_{\Gamma_e} P_{\gamma\alpha} \Phi^i \, \mathrm{d}\Gamma = ⨍_{-1}^{+1} P_{\gamma\alpha} \Phi^i(\xi) J(\xi) \, \mathrm{d}\xi \tag{43}$$

where Γ_e is the boundary of the singular element, Φ^i is the element shape function corresponding to the node i in the element under consideration and J is the Jacobian of the transformation from the x_α co-ordinate system to the local co-ordinate system ξ (i.e. $\mathrm{d}\Gamma = J(\xi)\,\mathrm{d}\xi$). To deal with this strong singularity, consider the Taylor series expansion about the singular point ξ_0 as follows:

$$\begin{aligned} \Phi^i(\xi) &= \Phi^i(\xi_0) + \Phi^{i\prime}(\xi_0)\delta\xi + \tfrac{1}{2}\Phi^{i\prime\prime}(\xi_0)\delta\xi^2 + \cdots \\ J(\xi) &= J(\xi_0) + J'(\xi_0)\delta\xi + \tfrac{1}{2}J''(\xi_0)\delta\xi^2 + \cdots \\ x_\alpha(\xi) &= x_\alpha(\xi_0) + x'_\alpha(\xi_0)\delta\xi + \tfrac{1}{2}x''_\alpha(\xi_0)\delta\xi^2 + \cdots \end{aligned} \tag{44}$$

in which

$$x_\alpha(\xi) = \sum_{i=1}^{3} M^i(\xi) x_\alpha^i \tag{45}$$

where, x_α^i is the co-ordinate x_α of the nodal point i, M^i is the corresponding geometric shape function and $(\cdot)'$ denotes the derivatives of $(\cdot)$ with respect to ξ and $\delta\xi = \xi - \xi_0$. Using the last relationship, the following expressions can be written for quadratic elements:[19]

$$\begin{aligned} r_\alpha &= x'_\alpha(\xi_0)\delta\xi + \tfrac{1}{2}x''_\alpha(\xi_0)\delta\xi^2 \\ r &= |\delta\xi| \sqrt{d_0 + d_1\delta\xi + d_2\delta\xi^2} \end{aligned} \tag{46}$$

where

$$\begin{aligned} d_0 &= x'_\alpha(\xi_0)x'_\alpha(\xi_0) \\ d_1 &= x'_\alpha(\xi_0)x''_\alpha(\xi_0) \\ d_2 &= \tfrac{1}{4}x''_\alpha(\xi_0)x''_\alpha(\xi_0) \end{aligned} \tag{47}$$

Noting that $\Phi^i(\xi) = 1$ at the collocation point, and using the expressions of the singular terms, equation (43) can be written in the following form:

$$\fint_{\Gamma_e} P_{\gamma\alpha}\Phi^i \,\mathrm{d}\Gamma = \int_{-1}^{+1} [P_{\gamma\alpha}\Phi^i(\xi)J(\xi) - S^P_{\gamma\alpha}(\xi_0)]\,\mathrm{d}\xi + \fint_{-1}^{+1} S^P_{\gamma\alpha}(\xi_0)\,\mathrm{d}\xi \tag{48}$$

and

$$S^P_{\gamma\alpha}(\xi_0) = \frac{1-\nu}{4\pi} J(\xi_0) \frac{x'_\gamma(\xi_0)n_\alpha(\xi_0) - x'_\alpha(\xi_0)n_\gamma(\xi_0)}{\delta\xi(d_0 + d_1\delta\xi + d_2\delta\xi^2)} \tag{49}$$

where $S^P_{\gamma\alpha}$ is the isolated singular term. As can be seen, the first integral on the right-hand side of equation (48) is not singular and it can be evaluated using a Gauss–Legendre scheme, whereas the second integral is singular and can be easily evaluated analytically.

9. INTERNAL POINT FUNCTIONS

After solving the problem to obtain the boundary unknowns the rotation and deflection at any internal point $\mathbf{X}'$ can be evaluated using the integral equation (36) with $c_{ij} = 1$. The equivalent boundary integrals for the body force domain integral in equation (40) can also be used, but with $\Delta\varphi = 2\pi$ as described in Reference 17.

The moment and shear stresses at the internal point $\mathbf{X}'$ can be evaluated by differentiating equation (36) with respect to the co-ordinate of the source point and substituting into equation (1) to give

$$M_{\alpha\beta}(\mathbf{X}') = \int_\Gamma U_{\alpha\beta j}(\mathbf{X}',\mathbf{x})p_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma P_{\alpha\beta j}(\mathbf{X}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) + \int_\Omega U_{\alpha\beta 3}(\mathbf{X}',\mathbf{X})q(\mathbf{X})\,\mathrm{d}\Omega(\mathbf{X}) \tag{50}$$

$$Q_\beta(\mathbf{X}') = \int_\Gamma U_{3\beta j}(\mathbf{X}',\mathbf{x})p_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma P_{3\beta j}(\mathbf{X}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) + \int_\Omega U^*_{3\beta 3}(\mathbf{X}',\mathbf{X})q(\mathbf{X})\,\mathrm{d}\Omega(\mathbf{X}) \tag{51}$$

where

$$\begin{aligned} U_{ijk} &= U_{ijk}^{\mathscr{A}} + U_{ijk}^{\mathscr{B}} \\ P_{ijk} &= P_{ijk}^{\mathscr{A}} + P_{ijk}^{\mathscr{B}} \end{aligned} \tag{52}$$

in which

$$\begin{aligned} U_{\alpha\beta j}^{\mathscr{A},\mathscr{B}} &= \frac{D(1-v)}{2}\left[U_{\alpha j,\beta}^{\mathscr{A},\mathscr{B}} + U_{\beta j,\alpha}^{\mathscr{A},\mathscr{B}} + \frac{2v}{1-v}U_{\gamma j,\gamma}^{\mathscr{A},\mathscr{B}}\delta_{\alpha\beta}\right] \\ U_{3\beta j}^{\mathscr{A},\mathscr{B}} &= \frac{D(1-v)\lambda^2}{2}[U_{\beta j}^{\mathscr{A},\mathscr{B}} + U_{3j,\beta}^{\mathscr{A},\mathscr{B}}] \end{aligned} \tag{53}$$

$$\begin{aligned} P_{\alpha\beta j}^{\mathscr{A},\mathscr{B}} &= \frac{D(1-v)}{2}\left[P_{\alpha j,\beta}^{\mathscr{A},\mathscr{B}} + P_{\beta j,\alpha}^{\mathscr{A},\mathscr{B}} + \frac{2v}{1-v}P_{\gamma j,\gamma}^{\mathscr{A},\mathscr{B}}\delta_{\alpha\beta}\right] \\ P_{3\beta j}^{\mathscr{A},\mathscr{B}} &= \frac{D(1-v)\lambda^2}{2}[P_{\beta j}^{\mathscr{A},\mathscr{B}} + P_{3j,\beta}^{\mathscr{A},\mathscr{B}}] \end{aligned} \tag{54}$$

The expressions for the kernels U_{ijk} and P_{ijk} are given in Appendix II.

Consequently, the domain terms in equations (50) and (51) can be replaced for a uniform domain load q by boundary integrals for any internal collocation point $\mathbf{X}'$, as

$$\int_{\Omega} U_{i\beta 3}(\mathbf{X}',\mathbf{X})q\,\mathrm{d}\Omega(\mathbf{X}) = q\int_{\Gamma} W_{i\beta}(\mathbf{X}',\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) \tag{55}$$

where

$$\begin{aligned} W_{\alpha\beta} &= \frac{D(1-v)}{2}\left[V_{\alpha,n\beta} + V_{\beta,n\alpha} + \frac{2v}{1-v}V_{\gamma,n\gamma}\delta_{\alpha\beta}\right] \\ W_{3\beta} &= \frac{D(1-v)\lambda^2}{2}[V_{\beta,n} + V_{3,n\beta}] \end{aligned} \tag{56}$$

The full expressions for $W_{i\beta}$ are given in Appendix I.

10. EXAMPLES

In this section, three numerical examples will be presented to demonstrate the accuracy and the efficiency of the present formulation. The examples will cover different types of boundary conditions. Throughout the examples, the normalized values are defined as follows:

$$\begin{aligned} &\overline{k_{\mathrm{f}}} = \frac{k_{\mathrm{f}}a^4}{D}, \quad \overline{w} = \frac{wD}{qa^4}, \quad \overline{M} = \frac{M}{qa^2}, \\ &\overline{Q} = \frac{Q}{qa^2} \quad \text{and} \quad \overline{\sigma} = \frac{\sigma}{q} \end{aligned} \tag{57}$$

to represent the normalized value of the modulus of subgrade reaction, deflection, moment, shear and stress, respectively.

Table I. Results for the clamped circular plate*

$\overline{k_f}$	h/a	$\overline{w_c}^{12}$	$\overline{w_c}$, present	$\overline{M_c}^{12}$	$\overline{M_c}$, present	$\overline{M_b}^{12}$	$\overline{M_b}$, present
	0·1	1·343468	1·347459	6·482927	6·524833	−10·821060	−10·789856
20	0·2	1·479259	1·483778	6·288024	6·329286	−10·566110	−10·515160
	0·3	1·689809	1·695230	5·984246	6·024738	−10·167780	−10·118336
	0·1	0·778565	0·780093	3·347978	3·377756	−7·336517	−7·285372
100	0·2	0·809990	0·811805	3·028080	3·057687	−6·822058	−6·774834
	0·3	0·850876	0·853225	2·593989	2·623383	−6·110953	−6·072877
	0·1	0·499372	0·499998	1·846688	1·874355	−5·538449	−5·491308
200	0·2	0·503068	0·504016	1·595467	1·623412	−4·999870	−4·960411
	0·3	0·506362	0·507894	1·282160	1·310455	−4·304075	−4·274937

*All of the normalized values are multiplied by 100

Table II. Limiting analysis for the clamped circular plate (Case 3)

κ	$\overline{w_c} \times 10^4$	$\overline{M_c} \times 100$	$\overline{M_b} \times 100$
0·00	220·526580	8·510521	−12·114212
0·01	214·871084	8·057708	−12·366922
0·03	203·675067	7·445186	−11·727621
0·05	179·723069	6·445598	−10·600387
0·10	113·959353	3·761325	−7·4795290

10.1. Clamped circular plate

In this example, a circular plate of a radius a is considered. Poisson's ratio is taken as 0·3. A uniform domain load of density q is applied over the plate domain. The following parametric studies are carried out:

1. A comparison between the present formulation using 16 quadratic boundary elements with the results obtained from Reference 12 for Case (3) using 32 linear boundary elements and cell integration is shown in Table I. The value of the central point deflection and moment and the boundary moment are considered in the comparison. Different values of $\overline{k_f}$ and plate thickness are considered. It can be seen that quadratic boundary elements represent the plate behaviour with less elements than the linear elements.
2. The second investigation studies the behaviour of the plate as the soil becomes weaker. The thickness of the plate is fixed as $0{\cdot}3a$ and the values of k_f are chosen to give $\kappa = 0{\cdot}1, 0{\cdot}05, 0{\cdot}03, 0{\cdot}01$. Table II shows the behaviour of the central point deflection and moment, and the boundary moment against the value of κ. The results at $\kappa = 0$ are obtained from the BEM analysis of Reissner plate under bending by the authors in Reference 9.

   As can be seen, the results of the Reissner plates under bending represent the limiting values for the analysis of a plate on an elastic foundation. It has to be noted that a disturbance is observed in the boundary moment when $\kappa = 0{\cdot}01$. That is mainly due to $\psi = 0{\cdot}7804$ which is very near to the limiting value $\pi/4$ which results in some inaccuracy in the computation of Hankel functions in the kernels of Case 3 ($\kappa < 1$).

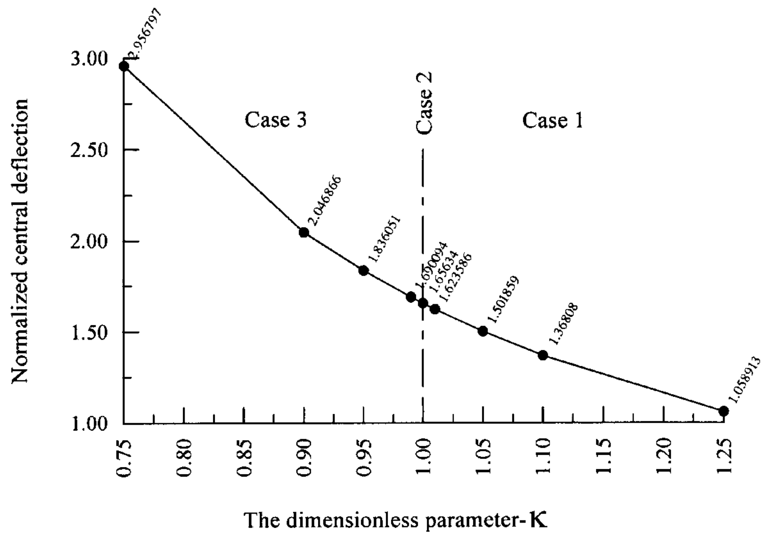

Figure 2. $\kappa - \overline{w_c} \times 10^4$ curve for the clamped circular plate

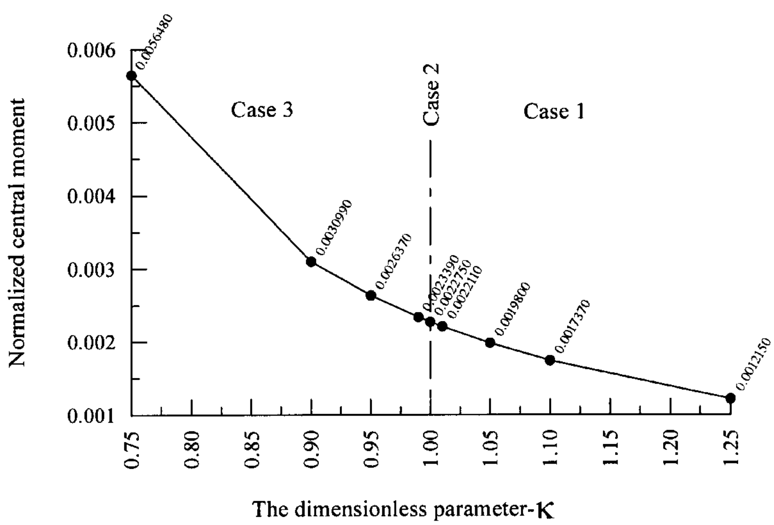

Figure 3. $\kappa - \overline{M_c} \times 100$ curve for the clamped circular plate

3. Here, the former problem is considered, with chosen values of k_f to give $0{\cdot}75 \leqslant \kappa \leqslant 1{\cdot}25$. Figures 2, 3 and 4 present the variation of $\overline{w_c} \times 10^4$, $\overline{M_c} \times 100$ and $\overline{M_b} \times 100$ against κ, respectively. From these figures, it can be seen that the agreement between the three cases is excellent, particularly in the vicinity of $\kappa = 0{\cdot}99, 1{\cdot}00, 1{\cdot}01$.

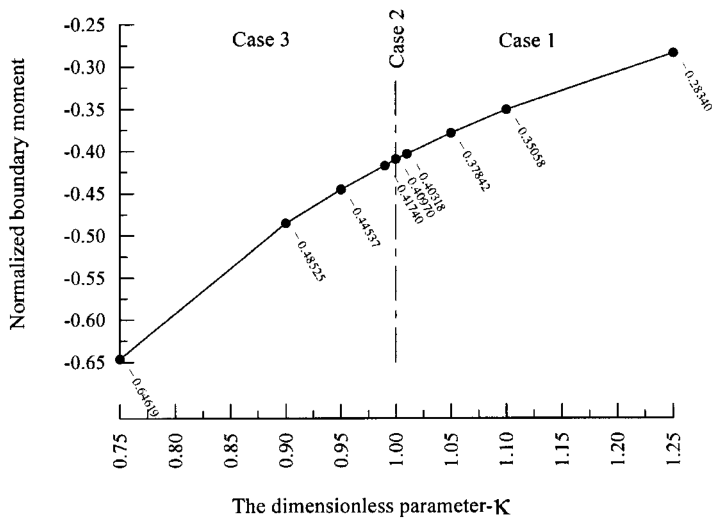

Figure 4. $\kappa - \overline{M_b} \times 100$ curve for the clamped circular plate

Table III. Results for the simply supported plate results

h/a	Result of	$\overline{w_c} \times 100$	$\overline{M_c} \times 100$
0·05	Reference 12, 16 linear elements	0·262126	2·935155
	Reference 12, 32 linear elements	0·265802	2·962147
	Present 16 quad. elements	0·265061	2·960994
	Series sol.[10] 100 term	0·267007	—
0·10	Reference 12, 16 linear elements	0·268455	2·890909
	Reference 12, 32 linear elements	0·271712	2·915889
	Present 16 quad. elements	0·271611	2·918182
	Series sol.[10] 100 term	0·271954	—
0·20	Reference 12, 16 linear elements	0·290886	2·721548
	Reference 12, 32 linear elements	0·293299	2·742774
	Present 16 quad. elements	0·293824	2·749050
	Series sol.[10] 100 term	0·290391	—
0·30	Reference 12, 16 linear elements	0·321824	2·474650
	Reference 12, 32 linear elements	0·323517	2·492709
	Present 16 quad. elements	0·324175	2·500181
	Series sol.[10] 100 term	0·316960	—

10.2. Simply supported square plate

A simply supported square plate of side length a is considered in this example. Poisson's ratio is taken to be 0·3. A uniform load q is applied over all of the plate domain. The plate is analysed using 16 boundary elements. The following parametric studies are carried out:

1. A comparison between the results obtained by Wang *et al.*[12] using 16 and 32 linear elements, present solution and the series solution[10] (see item 3 of this example) for a different plate

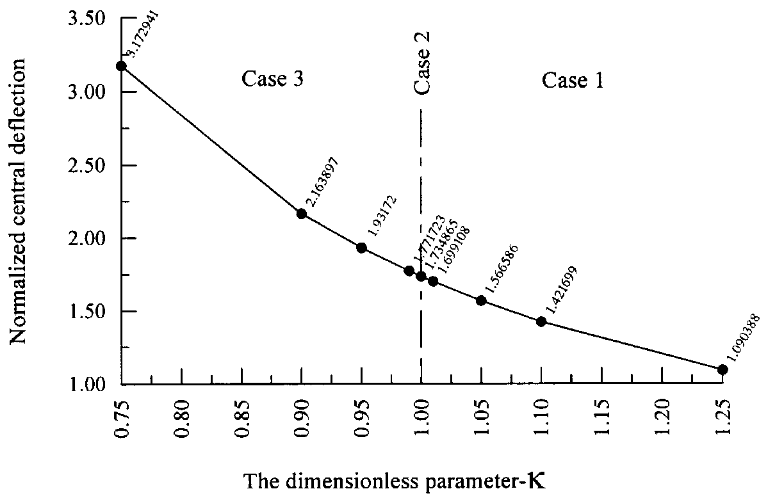

Figure 5. $\kappa - \overline{w_c} \times 10^4$ curve for the simply supported square plate

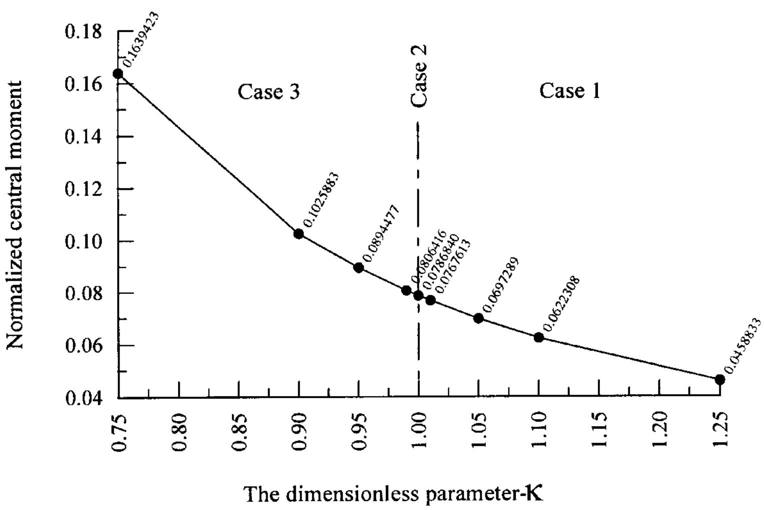

Figure 6. $\kappa - \overline{M_c} \times 100$ curve for the simply supported square plate

thickness is shown in Table III. The results obtained are in good agreement with those reported in Reference 12.

2. Here, the former problem is considered, with chosen values of k_f to give $0{\cdot}75 \leqslant \kappa \leqslant 1{\cdot}25$. The plate thickness is fixed as $0{\cdot}3a$. Figures 5 and 6 present the variation of $\overline{w_c} \times 10^4$ and $\overline{M_c} \times 100$ against κ. From these figures, it can be seen that the agreement between the three cases is excellent, particularly in the vicinity of $\kappa = 0{\cdot}99, 1{\cdot}00, 1{\cdot}01$. $\overline{k_f}$ is set equal 200.
3. A comparison was made with the series solution for the deflection and the shear given by Frederic.[10] The same plate is considered with thickness $0{\cdot}3a$. Figures 7 and 8 show comparison

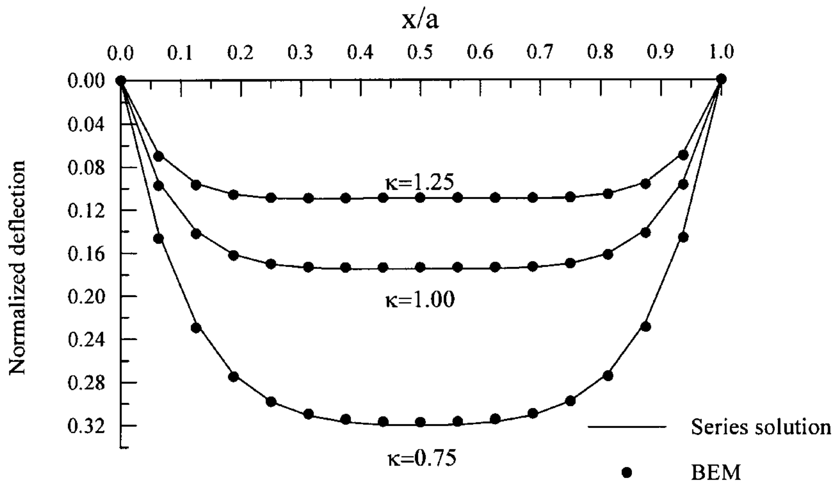

Figure 7. $\overline{w} \times 10^3$ distribution along the central line for the simply supported square plate

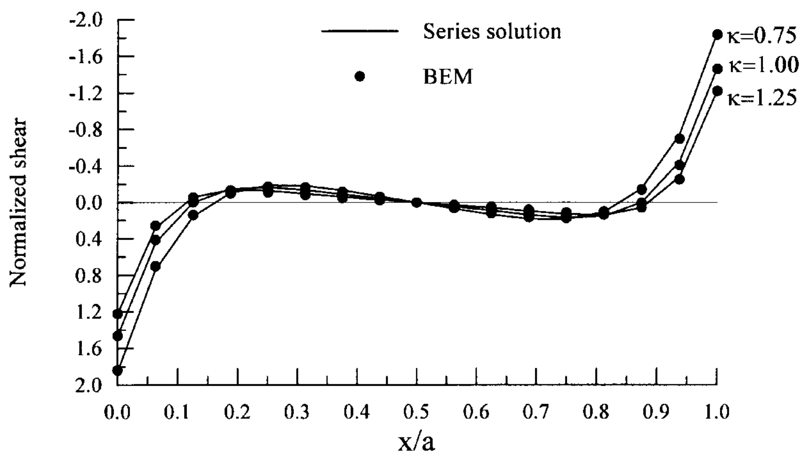

Figure 8. $\overline{Q} \times 100$ distribution along the central line for the simply supported square plate

between the normalized deflection $\overline{w} \times 10^3$ and the normalized shear $\overline{Q} \times 100$ along the central line $y = \frac{1}{2}a$ for the three cases ($\kappa = 0{\cdot}75, 1{\cdot}00, 1{\cdot}25$). As can be seen good agreement between the BEM solution and the series solution with 100 term is obtained.

10.3. Free edge plate

In this example a square plate of side length a and thickness $h = 0{\cdot}4a$ is considered. The plate has the free edge boundary condition along the four sides. Poisson's ratio is taken as 0·35. The plate resting on a Winkler foundation with modulus of subgrade reaction was selected to give $\kappa = 1{\cdot}25, 1{\cdot}00, 0{\cdot}75$ to represent cases 1, 2 and 3, respectively.

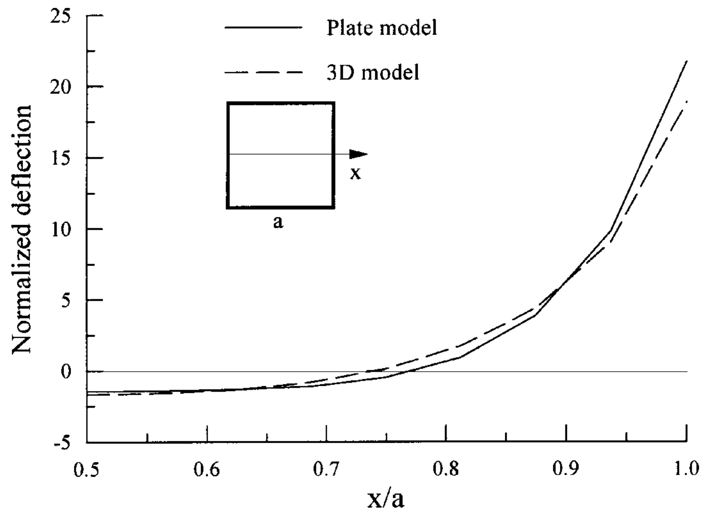

Figure 9. $\overline{w} \times 10^4$ distribution along the x-axis for the free edge square plate (Case 1: $\kappa = 1{\cdot}25$)

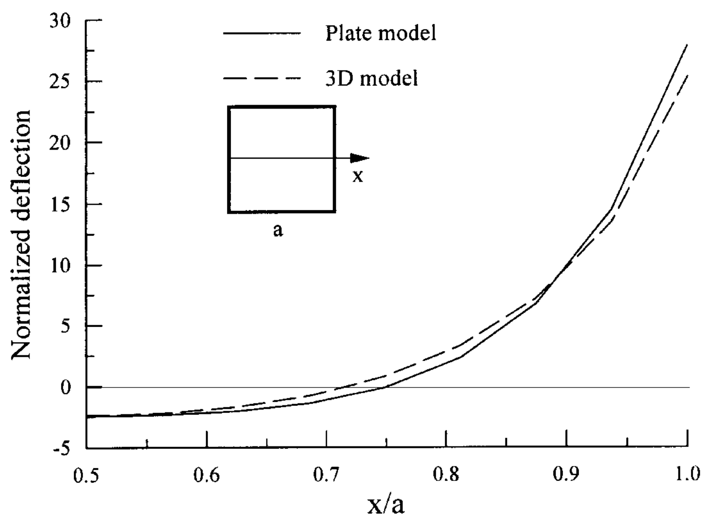

Figure 10. $\overline{w} \times 10^4$ distribution along the x-axis for the free edge square plate (Case 2: $\kappa = 1{\cdot}00$)

The BEM analysis was carried out using the present formulation and using a three-dimensional (3D) model. The plate bending model employed 16 boundary elements, whereas a mesh of $8 \times 8 \times 4$ quadratic boundary elements (9 noded-element) is considered in the 3-D analysis. The Winkler foundation model is represented in the 3-D analysis by spring boundary condition applied over the top and the bottom surfaces of the plate to minimize the plate deformation in the x_3 direction. The load is applied in the form of constant tractions qh along the boundary in the plate analysis, and as constant uniform shear load of intensity q in the x_3 direction over the four side surfaces of the plate.

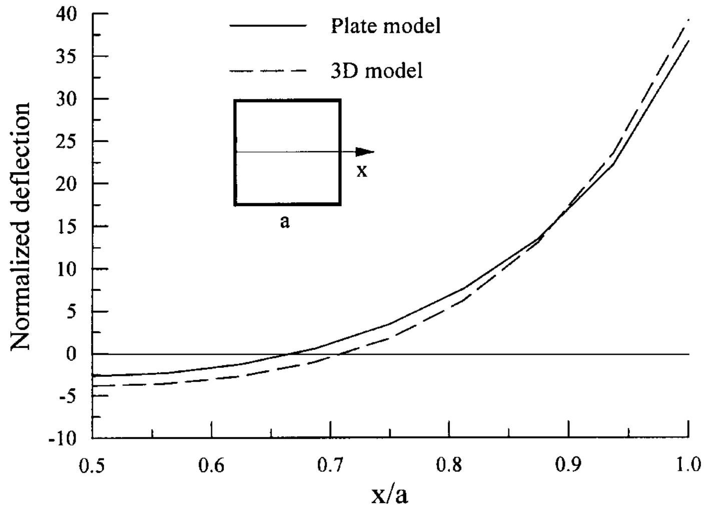

Figure 11. $\overline{w} \times 10^4$ distribution along the x-axis for the free edge square plate (Case 3: $\kappa = 0{\cdot}75$)

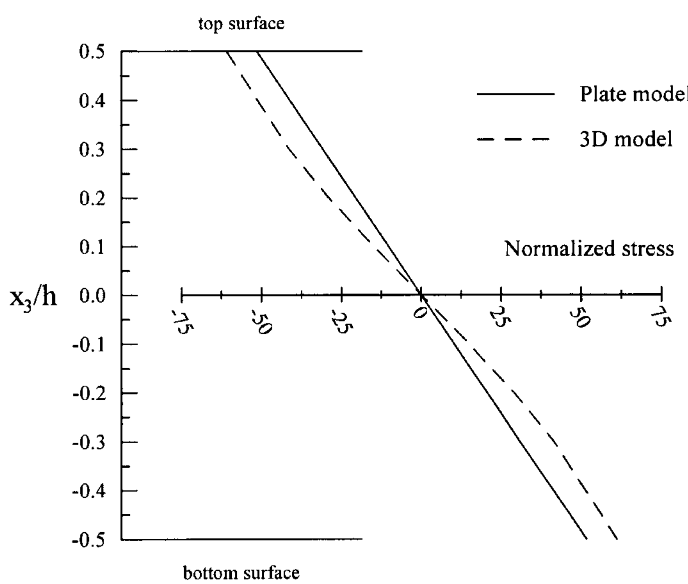

Figure 12. $\overline{\sigma_c} \times 100$ distribution under the central point for the free edge square plate (Case 1: $\kappa = 1{\cdot}25$)

Figures 9–11 show the deflection along the central axis x (from $x/a = 0{\cdot}5$ to $x/a = 1$) for the cases 1, 2 and 3, respectively. Figures 12–14 show the normal stress σ_c inside the plate along vertical line through the plate under its central point. The computed normal stress is based on the following relationship given by Reissner:[8]

$$\sigma_c = \frac{12x_3}{h^3} M_c \tag{58}$$

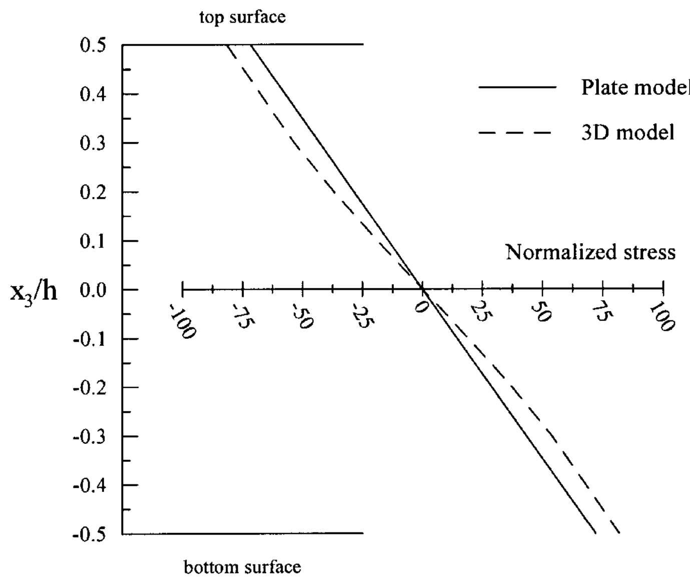

Figure 13. $\overline{\sigma_c} \times 100$ distribution under the central point for the free edge square plate (Case 2: $\kappa = 1{\cdot}00$)

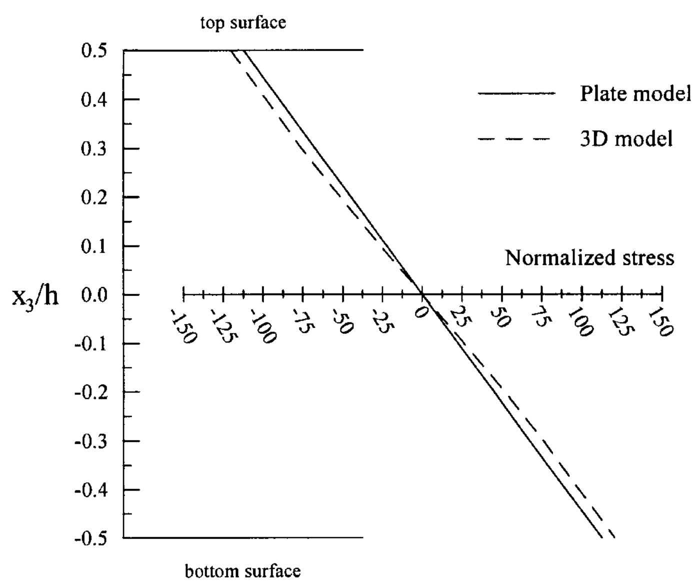

Figure 14. $\overline{\sigma_c} \times 100$ distribution under the central point for the free edge square plate (Case 3: $\kappa = 0{\cdot}75$)

As can be seen from Figures 9 and 14, the results are in good agreement with the 3-D solution. It is worth noting that the stresses are slightly different in the two models, because the 3-D solution is affected by the in plane deformation of the plate which becomes considerable as the soil strength increases by increasing κ. So that, the difference between the two solutions is higher in case 1 ($\kappa = 1{\cdot}25$) than in case 2 ($\kappa = 1{\cdot}00$) and case 3 ($\kappa = 0{\cdot}75$).

11. CONCLUSIONS

The fundamental solution of Reissner plates resting on a Winkler foundation was derived in this paper. The fundamental solution has three different cases depending on the plate and the foundation constants. One case was derived, previously, by Wang *et al.*[12] and the other two cases were derived in this work. It was shown that the fundamental solution contains singular terms of $O(\ln r)$ and $O(1/r)$. In the present work the body force integral was performed using equivalent boundary integrals. Quadratic elements were used to model the boundary geometry and unknowns. The integrals with a strong singularity were evaluated using a semi-analytical process via a Taylor series expansion. Three different numerical examples were solved including different types of boundary conditions. The results were compared to other published results, analytical results, and to a 3-D solution. The results are accurate and compatible between each other (for the different cases of the fundamental solution). It was also shown that, from the computational point of view, modelling moderately thick plates on an elastic foundation using the present model is more suitable than using a 3-D model.

APPENDIX I

Expressions for $V_{i,n}$ and $W_{i\beta}$

The expression for $V_{i,n}$ is given as follows.[17]

Case 1 ($\kappa > 1$):

$$V_{\alpha,n} = \frac{-\ell^2}{4\pi DS}\left[\frac{\Upsilon_{-1}}{r}(n_\alpha - 2r_{,\alpha}r_{,n}) - \Upsilon_0 r_{,\alpha}r_{,n}\right]$$
$$V_{3,n} = \frac{-\ell^2 r_{,n}}{4\pi DS}\left[\frac{-2}{(1-v)\lambda^2}\Upsilon_1 + \Upsilon_{-1}\right] \tag{59}$$

Case 2 ($\kappa = 1$):

$$V_{\alpha,n} = \frac{-\ell^2}{4\pi D}\left[A_1\left(\frac{r}{\ell}\right)(n_\alpha - 2r_{,\alpha}r_{,n}) - \left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right)r_{,\alpha}r_{,n}\right]$$
$$V_{3,n} = \frac{-r_{,n}\ell}{4\pi D(1-v)\lambda^2}\left[4K_1\left(\frac{r}{\ell}\right) + \ell^2\lambda^2\overline{\alpha}\left(2K_1\left(\frac{r}{\ell}\right) + \left(\frac{r}{\ell}\right)K_0\left(\frac{r}{\ell}\right)\right)\right] \tag{60}$$

Case 3 ($\kappa < 1$):

$$V_{\alpha,n} = \frac{-\ell^2}{8D\sin 2\psi}\left[\frac{\Psi_{-1}}{r}(n_\alpha - 2r_{,\alpha}r_{,n}) + \Psi_0 r_{,\alpha}r_{,n}\right]$$
$$V_{3,n} = \frac{-\ell^2 r_{,n}}{8D\sin 2\psi}\left[\frac{2}{(1-v)\lambda^2}\Psi_1 + \Psi_{-1}\right] \tag{61}$$

The expression for $W_{i\beta}$ is given as follows.[17]

Case 1 ($\kappa > 1$):

$$W_{\alpha\beta} = \frac{\ell^2}{4\pi S}\left\{ r_{,n}\Upsilon_1[\nu\delta_{\alpha\beta} + (1-\nu)r_{,\alpha}r_{,\beta}] + \frac{1-\nu}{r}\left[\frac{2\Upsilon_{-1}}{r} + \Upsilon_0\right](4r_{,\alpha}r_{,\beta}r_{,n} - \delta_{\alpha\beta}r_{,n} - n_\beta r_{,\alpha} - n_\alpha r_{,\beta})\right\} \quad (62)$$

$$W_{3\beta} = \frac{\ell^2}{4\pi S}\left\{ \Upsilon_2 r_{,\beta}r_{,n} - \frac{\Upsilon_1}{r}[n_\beta - 2r_{,\beta}r_{,n}]\right\}$$

Case 2 ($\kappa = 1$):

$$W_{\alpha\beta} = \frac{\ell(1-\nu)}{4\pi}\left\{ \left(\frac{r}{\ell}\right) K_0\left(\frac{r}{\ell}\right)\left[\frac{\nu}{1-\nu}\delta_{\alpha\beta}r_{,n} + r_{,\alpha}r_{,\beta}r_{,n}\right] + \left(\frac{\ell}{r}\right)\left[2A_1\left(\frac{r}{\ell}\right) + \left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right)\right](4r_{,\alpha}r_{,\beta}r_{,n} - \delta_{\alpha\beta}r_{,n} - n_\beta r_{,\alpha} - n_\alpha r_{,\beta})\right\} \quad (63)$$

$$W_{3\beta} = \frac{1}{4\pi}\left\{ \left(\frac{r}{\ell}\right) K_1\left(\frac{r}{\ell}\right) r_{,\beta}r_{,n} - K_0\left(\frac{r}{\ell}\right) n_\beta\right\}$$

Case 3 ($\kappa < 1$):

$$W_{\alpha\beta} = \frac{-\ell^2}{8\sin 2\psi}\left\{ r_{,n}\Psi_1[\nu\delta_{\alpha\beta} + (1-\nu)r_{,\alpha}r_{,\beta}] - \frac{1-\nu}{r}\left[\frac{2\Psi_{-1}}{r} - \Psi_0\right](4r_{,\alpha}r_{,\beta}r_{,n} - \delta_{\alpha\beta}r_{,n} - n_\beta r_{,\alpha} - n_\alpha r_{,\beta})\right\} \quad (64)$$

$$W_{3\beta} = \frac{\ell^2}{8\sin 2\psi}\left\{ \Psi_2 r_{,\beta}r_{,n} + \frac{\Psi_1}{r}[n_\beta - 2r_{,\beta}r_{,n}]\right\}$$

APPENDIX II

Expressions for U_{ijk} and P_{ijk}

The expressions for the kernels $U^{\mathscr{B}}_{\alpha\beta j}$ and $P^{\mathscr{B}}_{\alpha\beta j}$ do not depend on κ and are given by

$$\begin{aligned} U^{\mathscr{B}}_{\alpha\beta\gamma} &= \frac{-1}{2\pi r}\{zK_1(z)[2r_{,\alpha}r_{,\beta}r_{,\gamma} - \delta_{\alpha\gamma}r_{,\beta} - \delta_{\gamma\beta}r_{,\alpha}] \\ &\quad + 2A_1(z)[4r_{,\alpha}r_{,\beta}r_{,\gamma} - \delta_{\alpha\gamma}r_{,\beta} - \delta_{\gamma\beta}r_{,\alpha} - \delta_{\alpha\beta}r_{,\gamma}]\} \\ U^{\mathscr{B}}_{3\beta\alpha} &= \frac{\lambda^2}{2\pi}[B_1(z)\delta_{\alpha\beta} - A_1(z)r_{,\alpha}r_{,\beta}] \\ U^{\mathscr{B}}_{\alpha\beta 3} &= 0 \\ U^{\mathscr{B}}_{3\beta 3} &= 0 \end{aligned} \quad (65)$$

$$
\begin{aligned}
P^{\mathscr{B}}_{\alpha\beta\gamma} = {} & \frac{D(1-v)}{4\pi r^2}\{z^2K_0(z)(4r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} - \delta_{\alpha\gamma}r_{,\beta}r_{,n} - r_{,\gamma}r_{,\beta}n_\alpha - \delta_{\beta\gamma}r_{,\alpha}r_{,n} - n_\beta r_{,\alpha}r_{,\gamma}) \\
& + zK_1(z)(32r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} + 2\delta_{\gamma\beta}n_\alpha + 2\delta_{\gamma\alpha}n_\beta - 4n_\gamma r_{,\alpha}r_{,\beta} - 6n_\beta r_{,\alpha}r_{,\gamma} \\
& - 6n_\alpha r_{,\beta}r_{,\gamma} - 6\delta_{\beta\gamma}r_{,\alpha}r_{,n} - 6\delta_{\alpha\gamma}r_{,\beta}r_{,n} - 4\delta_{\alpha\beta}r_{,\gamma}r_{,n}) \\
& + 2A_1(z)(48r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} - 8\delta_{\alpha\gamma}r_{,\beta}r_{,n} - 8r_{,\gamma}r_{,\beta}n_\alpha - 8r_{,\alpha}r_{,\beta}n_\gamma - 8\delta_{\alpha\beta}r_{,\gamma}r_{,n} \\
& - 8\delta_{\beta\gamma}r_{,\alpha}r_{,n} - 8r_{,\alpha}r_{,\gamma}n_\beta + 2\delta_{\alpha\gamma}n_\beta + 2\delta_{\beta\gamma}n_\alpha + 2\delta_{\alpha\beta}n_\gamma)\} \\
P^{\mathscr{B}}_{3\beta\alpha} = {} & \frac{D(1-v)\lambda^2}{4\pi r}\{zK_1(z)[2r_{,\alpha}r_{,\beta}r_{,n} - n_\beta r_{,\alpha} - \delta_{\alpha\beta}r_{,n}] \\
& + 2A_1(z)[4r_{,\alpha}r_{,\beta}r_{,n} - \delta_{\alpha\beta}r_{,n} - n_\beta r_{,\alpha} - r_{,\beta}n_\alpha]\} \\
P^{\mathscr{B}}_{\alpha\beta3} = {} & \frac{D(1-v)\lambda^2}{4\pi r}\{zK_1(z)[n_\beta r_{,\alpha} + n_{,\alpha}r_{,\beta} - 2r_{,\alpha}r_{,\beta}r_{,n}] \\
& + 2A_1(z)[n_\beta r_{,\alpha} + n_\alpha r_{,\beta} + \delta_{\alpha\beta}r_{,n} - 4r_{,\alpha}r_{,\beta}r_{,n}]\} \\
P^{\mathscr{B}}_{3\beta3} = {} & \frac{D(1-v)\lambda^4}{4\pi}[B_1(z)n_\beta - A_1(z)r_{,\beta}r_{,n}]
\end{aligned}
\tag{66}
$$

The expressions for the kernels $U^{\mathscr{A}}_{\alpha\beta j}$ and $P^{\mathscr{A}}_{\alpha\beta j}$ depend on κ and are given as follows.

Case 1 ($\kappa > 1$):

$$
\begin{aligned}
U^{\mathscr{A}}_{\alpha\beta\gamma} = {} & -\frac{\ell^2}{4\pi\lambda^2 S}\left\{[\Upsilon_5 - \alpha\Upsilon_3]\left[2r_{,\alpha}r_{,\beta}r_{,\gamma} + \frac{2v}{1-v}\delta_{\alpha\beta}r_{,\gamma}\right]\right. \\
& \left. + \left[\frac{4}{r^2}(\Upsilon_3 - \alpha\Upsilon_1) + \frac{2}{r}(\Upsilon_4 - \alpha\Upsilon_2)\right][4r_{,\alpha}r_{,\beta}r_{,\gamma} - \delta_{\alpha\gamma}r_{,\beta} - \delta_{\gamma\beta}r_{,\alpha} - \delta_{\alpha\beta}r_{,\gamma}]\right\} \\
U^{\mathscr{A}}_{3\beta\alpha} = {} & +\frac{\ell^2}{4\pi S}\left\{\frac{1}{r}\left[\Upsilon_3 - \frac{k_{\mathrm{f}}}{C}\Upsilon_1\right](\delta_{\alpha\beta} - 2r_{,\alpha}r_{,\beta}) - \left[\Upsilon_4 - \frac{k_{\mathrm{f}}}{C}\Upsilon_2\right]r_{,\alpha}r_{,\beta}\right\} \\
U^{\mathscr{A}}_{\alpha\beta3} = {} & \frac{-\ell^2}{4\pi S}\left\{\Upsilon_2[(1-v)r_{,\alpha}r_{,\beta} + v\delta_{\alpha\beta}] - \frac{1-v}{r}\Upsilon_1[\delta_{\alpha\beta} - 2r_{,\alpha}r_{,\beta}]\right\} \\
U^{\mathscr{A}}_{3\beta3} = {} & \frac{-\ell^2}{4\pi S}\Upsilon_3 r_{,\beta}
\end{aligned}
\tag{67}
$$

$$P^{\mathscr{A}}_{\alpha\beta\gamma} = \frac{D(1-v)\ell^2}{8\pi\lambda^2 S}\Bigg\{[\Upsilon_6 - \alpha\Upsilon_4]\left[4r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} + \frac{4v}{1-v}\left(\delta_{\alpha\beta}r_{,\gamma}r_{,n} + n_\gamma r_{,\alpha}r_{,\beta} + \frac{v}{1-v}\delta_{\alpha\beta}n_\gamma\right)\right]$$

$$-\frac{1}{r}[\Upsilon_5 - \alpha\Upsilon_3]\Big[4n_\gamma r_{,\alpha}r_{,\beta} + 4n_\beta r_{,\alpha}r_{,\gamma} + 4n_\alpha r_{,\beta}r_{,\gamma} - 32r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n}$$

$$+4\delta_{\beta\gamma}r_{,\alpha}r_{,n} + 4\delta_{\alpha\gamma}r_{,\beta}r_{,n} + 4\delta_{\alpha\beta}r_{,\gamma}r_{,n} + \frac{4v}{1-v}(2\delta_{\alpha\beta}n_\gamma - 2n_\gamma r_{,\alpha}r_{,\beta} - 2\delta_{\alpha\beta}r_{,\gamma}r_{,n})\Big]$$

$$+\frac{4}{r^2}\left[\frac{2}{r}(\Upsilon_3 - \alpha\Upsilon_1) + (\Upsilon_4 - \alpha\Upsilon_2)\right](24r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} - 4\delta_{\alpha\gamma}r_{,\beta}r_{,n} - 4r_{,\gamma}r_{,\beta}n_\alpha$$

$$-4r_{,\alpha}r_{,\beta}n_\gamma - 4\delta_{\alpha\beta}r_{,\gamma}r_{,n} - 4\delta_{\beta\gamma}r_{,\alpha}r_{,n} - 4r_{,\alpha}r_{,\gamma}n_\beta + \delta_{\alpha\gamma}n_\beta + \delta_{\beta\gamma}n_\alpha + \delta_{\alpha\beta}n_\gamma)\Bigg\}$$

$$P^{\mathscr{A}}_{3\beta\alpha} = \frac{D(1-v)\ell^2}{8\pi S}\Bigg\{\left(\frac{4}{r^2}\left[\Upsilon_3 - \frac{k_f}{C}\Upsilon_1\right] + \frac{2}{r}\left[\Upsilon_4 - \frac{k_f}{C}\Upsilon_2\right]\right)$$

$$\times[4r_{,\alpha}r_{,\beta}r_{,n} - \delta_{\alpha\beta}r_{,n} - n_\beta r_{,\alpha} - r_{,\beta}n_\alpha] + \left[\Upsilon_5 - \frac{k_f}{C}\Upsilon_3\right]\left(2r_{,\alpha}r_{,\beta}r_{,n} + \frac{2v}{1-v}r_{,\beta}n_\alpha\right)\Bigg\}$$

$$P^{\mathscr{A}}_{\alpha\beta 3} = \frac{D(1-v)\ell^2}{4\pi S}\Bigg\{-\left[\Upsilon_5 - \frac{k_f}{C}\Upsilon_3\right]\left[r_{,\alpha}r_{,\beta}r_{,n} + \frac{v}{1-v}\delta_{\alpha\beta}r_{,n}\right]$$

$$+\left[\frac{2}{r^2}\left(\Upsilon_3 - \frac{k_f}{C}\Upsilon_1\right) + \frac{1}{r}\left(\Upsilon_4 - \frac{k_f}{C}\Upsilon_2\right)\right][n_\beta r_{,\alpha} + n_\alpha r_{,\beta} + \delta_{\alpha\beta}r_{,n} - 4r_{,\alpha}r_{,\beta}r_{,n}]\Bigg\}$$

$$P^{\mathscr{A}}_{3\beta 3} = \frac{D(1-v)\lambda^2\ell^2}{8\pi S}\frac{k_f}{C}\left\{\left(\Upsilon_2 + \frac{2}{r}\Upsilon_1\right)r_{,\beta}r_{,n} - \frac{\Upsilon_1}{r}n_\beta\right\} \tag{68}$$

Case 2 ($\kappa = 1$):

$$U^{\mathscr{A}}_{\alpha\beta\gamma} = \left[\frac{1}{\pi\lambda^2\ell^2 r}A_1\left(\frac{r}{\ell}\right) - \frac{\overline{\alpha}}{4\pi\ell}K_1\left(\frac{r}{\ell}\right)\right](4r_{,\alpha}r_{,\beta}r_{,\gamma} - \delta_{\alpha\gamma}r_{,\beta} - \delta_{\gamma\beta}r_{,\alpha} - \delta_{\alpha\beta}r_{,\gamma})$$

$$+\left[\frac{(1-v)}{2\pi\ell}K_1\left(\frac{r}{\ell}\right) - \frac{\overline{\alpha}}{4\pi\ell}\left(\frac{r}{\ell}\right)K_0\left(\frac{r}{\ell}\right)\right]\left(\frac{v}{1-v}\delta_{\alpha\beta}r_{,\gamma} + r_{,\alpha}r_{,\beta}r_{,\gamma}\right)$$

$$U^{\mathscr{A}}_{3\beta\alpha} = -\frac{1}{4\pi\ell^2}\left\{A_1\left(\frac{r}{\ell}\right)(\delta_{\alpha\beta} - 2r_{,\alpha}r_{,\beta}) - \left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right)r_{,\alpha}r_{,\beta}\right\} \tag{69}$$

$$U^{\mathscr{A}}_{\alpha\beta 3} = \frac{1}{4\pi}\left\{K_0\left(\frac{r}{\ell}\right)(1+v)\delta_{\alpha\beta} - \left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right)[v\delta_{\alpha\beta} + (1-v)r_{,\alpha}r_{,\beta}]\right\}$$

$$U^{\mathscr{A}}_{3\beta 3} = \frac{r_{,\beta}}{4\pi\ell}\left[2K_1\left(\frac{r}{\ell}\right) - \left(\frac{r}{\ell}\right)K_0\left(\frac{r}{\ell}\right)\right]$$

$$P^{\mathscr{A}}_{\alpha\beta\gamma} = \frac{D(1-v)}{4\pi\ell}\left\{\frac{1}{\ell(1-v)}\left[\frac{\overline{\alpha}}{1-v}\left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right) - 2K_0\left(\frac{r}{\ell}\right)\right]\right.$$
$$\times[v^2\delta_{\alpha\beta}n_\gamma + v(1-v)(\delta_{\alpha\beta}r_{,\gamma}r_{,n} + n_\gamma r_{,\alpha}r_{,\beta}) + (1-v)^2 r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n}]$$
$$+\frac{\overline{\alpha}}{\ell}K_0\left(\frac{r}{\ell}\right)[6r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} - n_\gamma r_{,\alpha}r_{,\beta} - n_\beta r_{,\alpha}r_{,\gamma} - n_\alpha r_{,\beta}r_{,\gamma} - \delta_{\gamma\beta}r_{,\alpha}r_{,n}$$
$$-\delta_{\alpha\gamma}r_{,\beta}r_{,n} - \delta_{\alpha\beta}r_{,\gamma}r_{,n}] - \frac{2v}{(1-v)^2}\frac{\overline{\alpha}}{\ell}K_0\left(\frac{r}{\ell}\right)n_\gamma\delta_{\alpha\beta}$$
$$+\left[\frac{\overline{\alpha}}{r}K_1\left(\frac{r}{\ell}\right) - \frac{4}{\ell\lambda^2 r^2}A_1\left(\frac{r}{\ell}\right)\right][\delta_{\gamma\beta}n_\alpha + \delta_{\alpha\gamma}n_\beta + \delta_{\alpha\beta}n_\gamma - 4n_\gamma r_{,\alpha}r_{,\beta} - 4n_\beta r_{,\alpha}r_{,\gamma}$$
$$-4n_\alpha r_{,\beta}r_{,\gamma} - 4\delta_{\gamma\beta}r_{,\alpha}r_{,n} - 4\delta_{\alpha\gamma}r_{,\beta}r_{,n} - 4\delta_{\alpha\beta}r_{,\gamma}r_{,n} + 24r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n}]$$
$$+\frac{2}{r}K_1\left(\frac{r}{\ell}\right)[(1-v)(n_\beta r_{,\alpha}r_{,\gamma} + n_\alpha r_{,\beta}r_{,\gamma} + n_\gamma r_{,\alpha}r_{,\beta} + \delta_{\gamma\beta}r_{,\alpha}r_{,n}$$
$$\left.+\delta_{\alpha\gamma}r_{,\beta}r_{,n} + \delta_{\alpha\beta}r_{,\gamma}r_{,n} - 8r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n}) + 2v(\delta_{\alpha\beta}n_\gamma - n_\gamma r_{,\alpha}r_{,\beta} - \delta_{\alpha\beta}r_{,\gamma}r_{,n}]\right\} \tag{70}$$

$$P^{\mathscr{A}}_{3\beta\alpha} = \frac{D(1-v)}{8\pi\ell}\left\{\left[\frac{4}{r\ell}A_1\left(\frac{r}{\ell}\right) + \left(\frac{k_{\rm f}}{C}\right)K_1\left(\frac{r}{\ell}\right)\right][\delta_{\alpha\beta}r_{,n} + n_\beta r_{,\alpha} + r_{,\beta}n_\alpha - 4r_{,\alpha}r_{,\beta}r_{,n}]\right.$$
$$\left.-\left(\frac{k_{\rm f}}{C}\right)\left(\frac{r}{\ell}\right)K_0\left(\frac{r}{\ell}\right)\left[\frac{v}{1-v}n_\alpha r_{,\beta} + r_{,\alpha}r_{,\beta}r_{,n}\right]\right\}$$

$$P^{\mathscr{A}}_{\alpha\beta3} = \frac{D(1-v)}{8\pi\ell^3}\left\{\left[4\left(\frac{\ell}{r}\right)A_1\left(\frac{r}{\ell}\right) + 2K_1\left(\frac{r}{\ell}\right)\right](4r_{,\alpha}r_{,\beta}r_{,n} - n_\beta r_{,\alpha} - n_\alpha r_{,\beta} - \delta_{\alpha\beta}r_{,n})\right.$$
$$\left.+2\left(\frac{r}{\ell}\right)K_0\left(\frac{r}{\ell}\right)\left[\frac{v}{1-v}\delta_{\alpha\beta}r_{,n} + r_{,\alpha}r_{,\beta}r_{,n}\right]\right\}$$

$$P^{\mathscr{A}}_{3\beta3} = \frac{D(1-v)\lambda^2}{8\pi}\left(\frac{k_{\rm f}}{C}\right)\left\{\left(\frac{r}{\ell}\right)K_1\left(\frac{r}{\ell}\right)r_{,\beta}r_{,n} - K_0\left(\frac{r}{\ell}\right)n_\beta\right\}$$

Case 3 ($\kappa < 1$):

$$U^{\mathscr{A}}_{\alpha\beta\gamma} = -\frac{\ell^2}{8\lambda^2\sin 2\psi}\left\{-[\Psi_5 + \alpha\Psi_3]\left[2r_{,\alpha}r_{,\beta}r_{,\gamma} + \frac{2v}{1-v}\delta_{\alpha\beta}r_{,\gamma}\right]\right.$$
$$\left.+\left[\frac{4}{r^2}(\Psi_3 + \alpha\Psi_1) - \frac{2}{r}(\Psi_4 + \alpha\Psi_2)\right][4r_{,\alpha}r_{,\beta}r_{,\gamma} - \delta_{\alpha\gamma}r_{,\beta} - \delta_{\gamma\beta}r_{,\alpha} - \delta_{\alpha\beta}r_{,\gamma}]\right\} \tag{71}$$

$$U^{\mathscr{A}}_{3\beta\alpha} = \frac{\ell^2}{8\sin 2\psi}\left\{\frac{1}{r}\left[\Psi_3 + \frac{k_{\rm f}}{C}\Psi_1\right](\delta_{\alpha\beta} - 2r_{,\alpha}r_{,\beta}) + \left[\Psi_4 + \frac{k_{\rm f}}{C}\Psi_2\right]r_{,\alpha}r_{,\beta}\right\}$$

$$U^{\mathscr{A}}_{\alpha\beta3} = \frac{-\ell^2}{8\sin 2\psi}\left\{\Psi_2[(1-v)r_{,\alpha}r_{,\beta} + v\delta_{\alpha\beta}] + \frac{1-v}{r}\Psi_1[\delta_{\alpha\beta} - 2r_{,\alpha}r_{,\beta}]\right\}$$

$$U^{\mathscr{A}}_{3\beta3} = \frac{-\ell^2}{8\sin 2\psi}\Psi_3 r_{,\beta}$$

$$
\begin{aligned}
P^{\mathscr{A}}_{\alpha\beta\gamma} = \frac{D(1-v)\ell^2}{16\lambda^2 \sin 2\psi} \Bigg\{ & [\Psi_6 + \alpha\Psi_4] \Bigg[4r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} \\
& + \frac{4v}{1-v}\left(\delta_{\alpha\beta}r_{,\gamma}r_{,n} + n_\gamma r_{,\alpha}r_{,\beta} + \frac{v}{1-v}\delta_{\alpha\beta}n_\gamma\right)\Bigg] \\
& + \frac{1}{r}[\Psi_5 + \alpha\Psi_3]\Bigg[4n_\gamma r_{,\alpha}r_{,\beta} + 4n_\beta r_{,\alpha}r_{,\gamma} + 4n_\alpha r_{,\beta}r_{,\gamma} - 32r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} \\
& + 4\delta_{\beta\gamma}r_{,\alpha}r_{,n} + 4\delta_{\alpha\gamma}r_{,\beta}r_{,n} + 4\delta_{\alpha\beta}r_{,\gamma}r_{,n} + \frac{4v}{1-v}(2\delta_{\alpha\beta}n_\gamma - 2n_\gamma r_{,\alpha}r_{,\beta} - 2\delta_{\alpha\beta}r_{,\gamma}r_{,n})\Bigg] \\
& + \frac{4}{r^2}\left[\frac{2}{r}(\Psi_3 + \alpha\Psi_1) - (\Psi_4 + \alpha\Psi_2)\right](24r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n} - 4\delta_{\alpha\gamma}r_{,\beta}r_{,n} - 4r_{,\gamma}r_{,\beta}n_\alpha \\
& - 4r_{,\alpha}r_{,\beta}n_\gamma - 4\delta_{\alpha\beta}r_{,\gamma}r_{,n} - 4\delta_{\beta\gamma}r_{,\alpha}r_{,n} - 4r_{,\alpha}r_{,\gamma}n_\beta + \delta_{\alpha\gamma}n_\beta + \delta_{\beta\gamma}n_\alpha + \delta_{\alpha\beta}n_\gamma)\Bigg\}
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
P^{\mathscr{A}}_{3\beta\alpha} = \frac{D(1-v)\ell^2}{16\sin 2\psi} \Bigg\{ & \left(\frac{4}{r^2}\left[\Psi_3 + \frac{k_f}{C}\Psi_1\right] - \frac{2}{r}\left[\Psi_4 + \frac{k_f}{C}\Psi_2\right]\right) \\
& \times [4r_{,\alpha}r_{,\beta}r_{,n} - \delta_{\alpha\beta}r_{,n} - n_\beta r_{,\alpha} - r_{,\beta}n_\alpha] - \left[\Psi_5 + \frac{k_f}{C}\Psi_3\right]\left(2r_{,\alpha}r_{,\beta}r_{,n} + \frac{2v}{1-v}r_{,\beta}n_\alpha\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
P^{\mathscr{A}}_{\alpha\beta 3} = \frac{D(1-v)\ell^2}{8\sin 2\psi} \Bigg\{ & \left[\Psi_5 + \frac{k_f}{C}\Psi_3\right]\left[r_{,\alpha}r_{,\beta}r_{,n} + \frac{v}{1-v}\delta_{\alpha\beta}r_{,n}\right] \\
& + \left[\frac{2}{r^2}(\Psi_3 + \alpha_1\Psi_1) - \frac{1}{r}(\Psi_4 + \alpha_1\Psi_2)\right][n_\beta r_{,\alpha} + n_\alpha r_{,\beta} + \delta_{\alpha\beta}r_{,n} - 4r_{,\alpha}r_{,\beta}r_{,n}]\Bigg\}
\end{aligned}
$$

$$
P^{\mathscr{A}}_{3\beta 3} = \frac{D(1-v)\lambda^2\ell^2}{16\sin 2\psi}\left(\frac{k_f}{C}\right)\left\{\left(\Psi_2 - \frac{2}{r}\Psi_1\right)r_{,\beta}r_{,n} + \frac{\Psi_1}{r}n_\beta\right\}
$$

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