Engineering Analysis with Boundary Elements 32 (2008) 152–161

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

www.elsevier.com/locate/enganabound

A relative quantity integral equation formulation for evaluation of boundary stress resultants in shear deformable plate bending problems

Youssef F. Rashed*,[1]

The British University in Egypt (BUE), El-Sherouk City, Postal No. 11837, P.O. Box 43, Egypt

Received 3 January 2007; accepted 2 July 2007
Available online 4 September 2007

Abstract

This paper presents new boundary element formulation for the shear deformable plate bending problems based on considering the relative displacement quantities. This can be done by subtracting the rigid body integral equations from the traditional displacement integral equations. The result is a new set of regularized integral equations, which could be used without any special integration considerations to compute the boundary generalized displacements. Using suitable constitutive relationships, new integral equations for computing boundary stress resultants are formulated. The behaviour of such equations are studied at smooth boundary points. Unlike the traditional formulation which contains kernels of $O(1/r^2)$, the newly derived equations are of $O(1/r)$, therefore they can be used to compute boundary values with the traditional quadratic continuous boundary elements. The main advantage of the present formulation is that it can be used to compute values anywhere within the plate domain or on the boundary, which gives the boundary element formulation a practical essence.

Keywords: Boundary element method; Boundary values; Plate bending; Relative displacements

1. Introduction

The boundary element method (BEM) [1] is a powerful tool in computational engineering. The method has several advantages, which make it attractive to be used in practical engineering. The main challenge for the BEM is how it will be used to serve in the solution and modelling of practical problems. The question: "Whether the BEM will be used in practice as the case of the finite element method?" is frequently repeated in BEM conferences or research meetings. In the last 10 years several theoretical BEM formulations have been implemented inside practical software packages (see for example the BEASY [1]). Among those are applications in fracture mechanics problems, cathodic protection applications, mechanical engineering, electro-magnetic applications, geo-technical problems, etc. The author in Ref. [2] has presented BEM formulation for solution of flat slab problems. In Ref. [3], he extended the formulation to solve building foundation problems. A serious problem [4,5] always faces the BEM solution is the near boundary and the boundary values, especially values for stress resultants (bending moments and shear forces). Such values are computed in an inaccurate manner in the vicinity of the boundary due to computation of singular (and hyper-singular) integrals of $O(1/r^2)$. From practical point of view this problem limits the accurate usage of the BEM within the engineering community. In addition, any graphical postprocessor for the results will give wrong overshooting values near and on the problem boundary. Moreover the application of the BEM in shape optimization problems will be limited; as such applications are dependent on computing the function derivatives on the boundaries.

In the literature several research works have been carried out concerning the plate bending theory (see for example [4–8]); however, in the next paragraphs only relevant work is considered. Researchers have considered several techniques to compute the boundary values in boundary elements. One of those techniques is the computation

*Tel.: +20 10 511 2949; fax: +20 2 2687 5889.
E-mail address: yrashed@bue.edu.eg
[1]On leave from: Department of Structural Engineering, Cairo University, Giza, Egypt.

0955-7997/$ - see front matter
doi:10.1016/j.enganabound.2007.07.003

using the hyper-singular integral equations on the boundary [4]. The idea of such technique is to consider the internal bending moment and shear force integral equations on the boundary. The final form of this formulation is a set of integral equations having singularity of $O(1/r^2)$. Despite the high accuracy of this technique, it is computationally expensive to compute. Moreover, fully discontinuous boundary elements have to be used to satisfy the continuity requirements (C^1) imposed by the formulation.

An alternative technique, which was considered to compute boundary values, is called the shape function derivative [5], which compute the missing tangential boundary value using the displacement derivative obtained from the assumed distribution of the boundary displacements (i.e. using the shape function derivative). Such a technique is approximate and needs fine boundary discretization together with using higher order elements (at least quadratic, in order to have linear distribution for the strains). Both of the former techniques were considered for elasticity problems by Aliabadi and Rooke [9] and for plate bending problems by Rashed [10].

In 1988, Kawahara and Kisu [11] derived new formulation for two-dimensional elasticity problems by formulating the integral equation in terms of the relative displacement quantities. The main advantage of this formulation is the new displacement integral equation being weakly singular and the internal stress integral equation having singularity of $O(1/r)$. Therefore such new integral equations can compute the boundary values easily. In 1992, Li and Obata [12] extended the formulation of Ref. [11] to three-dimensional problems. Since that time, to the author's best knowledge, such technique has never been applied to plate bending problems.

The purpose of this paper is to formulate the boundary integral equations for the shear deformable plate bending problems using the relative displacement quantities. This can be done by subtracting the rigid body integral equations from the traditional generalized displacement integral equations. Unlike elasticity problems, the rigid body displacements for plate bending problems generate coupled displacements. Therefore, it is expected that the present formulation will be more difficult and complicated compared to the elasticity formulations in Refs [11,12]. However, such complexity cannot be considered as disadvantage as the present formulation will be offered to practical design engineers as black box. Hence, new integral equations for computing boundary stress resultants are formulated to compute boundary values with the traditional quadratic continuous boundary elements. Example problem is presented to show the efficiency and the accuracy of the present formulation.

2. Traditional BEM for plate bending problems

In this section the boundary integral equation for thick plates according to Reissner [13] will be reviewed. Indicial notation will be used, in which Roman indices will vary from 1 to 3 and Greek indices will vary from 1 to 2. The stress-resultant generalized displacement relationships can be written as follows [13]:

$$M_{\alpha\beta}(\xi) = D\frac{1-v}{2}\left(u_{\alpha,\beta}(\xi) + u_{\beta,\alpha}(\xi) + \frac{2v}{1-v}u_{\gamma,\gamma}(\xi)\delta_{\alpha\beta}\right) + \frac{vq}{(1-v)\lambda^2}\delta_{\alpha\beta}, \tag{1}$$

$$Q_{3\beta}(\xi) = D\frac{1-v}{2}\lambda^2(u_\beta(\xi) + u_{3,\beta}(\xi)), \tag{2}$$

where $M_{\alpha\beta}(\xi)$ and $Q_{3\beta}(\xi)$ are the bending moments and shear force stress resultants, respectively, at point ξ, q is the uniform domain loading on the plate domain, D is the plate modulus of rigidity, λ is the shear factor, v is the Poisson's ratio, u_α denotes the rotation in two direction, u_3 denotes the deflection, the comma notation denotes derivatives according to the indicial notations and the symbol $\delta_{\alpha\beta}$ denotes the identity matrix. The corresponding integral equation formulation for internal point ξ can be formed as follows [8]:

$$u_i(\xi) + \int_{\Gamma(x)} T_{ij}(\xi, x)u_j(x)\,d\Gamma(x) = \int_{\Gamma(x)} U_{ij}(\xi, x)t_j(x)\,d\Gamma(x) + \int_{\Gamma(x)}\left[V_{i,n}(\xi, x) - \frac{v}{(1-v)\lambda^2}U_{in}(\xi, x)\right]q\,d\Gamma(x), \tag{3}$$

where $U_{ij}(\xi,x)$, $T_{ij}(\xi,x)$, $V_i(\xi,x)$ are the relevant fundamental solution kernels [8]. The notation ($_{,n}$) denotes the derivative with respect to the normal component at the boundary point x, whereas the subscript ($_n$) denotes the normal component of the kernel. The boundary generalized displacements and tractions are denoted by $u_j(x)$ and $t_j(x)$, respectively. Taking the point ξ to the plate boundary $\Gamma(x)$ it gives [8]

$$\frac{1}{2}u_i(\xi) + \mathbf{CPV}\int_{\Gamma(x)} T_{ij}(\xi, x)u_j(x)\,d\Gamma(x) = \int_{\Gamma(x)} U_{ij}(\xi, x)t_j(x)\,d\Gamma(x) + \int_{\Gamma(x)}\left[V_{i,n}(\xi, x) - \frac{v}{(1-v)\lambda^2}U_{i\alpha}(\xi, x)\right]q\,d\Gamma(x), \tag{4}$$

where the symbol **CPV** denotes that the integral is interpreted in Cauchy principal value sense.

3. Rigid body integral equations

Considering the rigid body motion of the plate, all boundary tractions are set to be zeros ($t_j(x) = 0$, $j = 1, 3$). Three cases have to be considered:

$$\text{when } u_1(\xi) = A \quad \text{then } u_1(x) = A,\ u_2(x) = 0, \text{ and } u_3(x) = [x_1(\xi) - x_1(x)]A, \tag{5}$$

when $u_2(\xi) = B$ then $u_1(\mathrm{x}) = 0$, $u_2(\mathrm{x}) = B$, and

$$u_3(\mathrm{x}) = [x_2(\xi) - x_2(\mathrm{x})]B, \tag{6}$$

when $u_3(\mathrm{x}) = C$ then $u_1(\mathrm{x}) = 0$, $u_2(\mathrm{x}) = 0$, and

$$u_3(\mathrm{x}) = C, \tag{7}$$

where, A, B, C are arbitrary constants, and x_1, x_2 are the Cartesian coordinates. Recalling Eqs. (3) and (4), Eqs. (5)–(7) can be written in the following form (considering $q = 0$):

$$u_i(\xi) + \int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x})u_k(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) = 0 \quad (\xi \text{ is an internal point}), \tag{8}$$

$$\frac{1}{2}u_i(\xi) + \mathbf{CPV}\int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x})u_k(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) = 0 \quad (\xi \text{ is a boundary point}), \tag{9}$$

where

$$D_{jk}(\xi,\mathrm{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1(\xi) - x_1(\mathrm{x}) & x_2(\xi) - x_2(\mathrm{x}) & 1 \end{bmatrix} \tag{10}$$

and

$$u_j(\xi) = u_j(\mathrm{x}) = \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = u_j(\mathrm{Q}), \tag{11}$$

where Q is any arbitrary boundary or internal point. It has to be noted that the matrix D_{jk} (which contains variable quantities that couples the coordinates of the source and the field points) is the main difference between the elasticity formulation and the present formulation for the plate bending problems.

4. Proposed relative displacement formulation

Considering ξ as internal point, subtracting Eq. (8) from Eq. (3), and taking into account that the rigid body generalized displacements are taken at an arbitrary point Q (recall Eq. (11)), it gives

$$\begin{aligned} u_i(\xi) - u_i(\mathrm{Q}) &+ \int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})(u_j(\mathrm{x}) - D_{jk}(\xi,\mathrm{x})u_k(\mathrm{Q}))\,\mathrm{d}\Gamma(\mathrm{x}) \\ &= \int_{\Gamma(\mathrm{x})} U_{ij}(\xi,\mathrm{x})t_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \int_{\Gamma(\mathrm{x})} \left[V_{i,n}(\xi,\mathrm{x}) - \frac{v}{(1-v)\lambda^2}U_{in}(\xi,\mathrm{x})\right] q\,\mathrm{d}\Gamma(\mathrm{x}). \end{aligned} \tag{12}$$

Eq. (12) is valid for any point Q. If Eq. (12) is considered as ξ is taken to the boundary, and considering Fig. 1 with the following relationships:

$$r_{,1} = n_1 = \cos\phi, \tag{13}$$

$$r_{,2} = n_2 = \sin\phi, \tag{14}$$

$$r = \varepsilon \quad \text{and} \quad r_{,n} = 1, \tag{15}$$

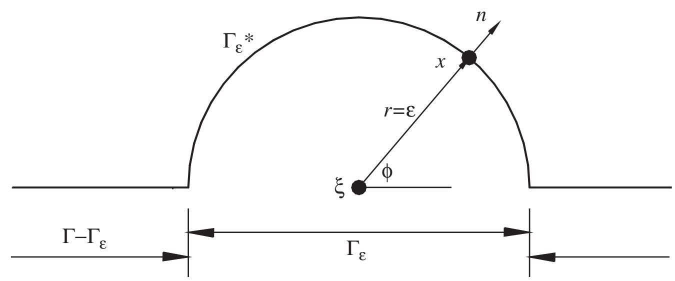

Fig. 1. The point ξ on smooth boundary.

$$\int_{\Gamma(\mathrm{x})} (\bullet)\,\mathrm{d}\Gamma(\mathrm{x}) = \lim_{\varepsilon\to 0}\left[\int_{\Gamma(\mathrm{x})-\Gamma_\varepsilon} (\bullet)\,\mathrm{d}\Gamma(\mathrm{x}) + \int_{\Gamma_\varepsilon^*} (\bullet)\,\mathrm{d}\Gamma(\mathrm{x})\right], \tag{16}$$

$$\int_{\Gamma_\varepsilon^*} (\bullet)\,\mathrm{d}\Gamma(\mathrm{x}) = \int_{\phi=0}^{\phi=\pi} (\bullet)\varepsilon\,\mathrm{d}\phi, \tag{17}$$

where $(\bullet)$ is any integrand. It can be seen that the first term of the integral on the LHS of Eq. (12) (recall the T_{ij} kernel is strong singular of $\mathrm{O}(1/r)$) will lead to the well-known $\frac{1}{2}$ jump term. Therefore, the following equation can be written:

$$\int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})u_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \Rightarrow \mathbf{CPV}\int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})u_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) + \frac{1}{2}u_i(\xi). \tag{18}$$

The matrix D_{jk} can be re-written in the following form (recall Eq. (10)):

$$D_{jk}(\xi,\mathrm{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1(\xi) - x_1(\mathrm{x}) & x_2(\xi) - x_2(\mathrm{x}) & 0 \end{bmatrix} = \delta_{jk} + (x_k(\xi) - x_k(\mathrm{x}))\delta_{j3}, \tag{19}$$

where $x_3(\xi) = x_3(\mathrm{x}) = 0$. Then the second term of the considered integral on the LHS of Eq. (12) can be written as follows (after considering the same steps as those for the first term) to give

$$\begin{aligned} \int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x})u_k(\mathrm{Q})\,\mathrm{d}\Gamma(\mathrm{x}) &= \int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})(\delta_{jk} + (x_k(\xi) - x_k(\mathrm{x}))\delta_{j3})u_k(\mathrm{Q})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &\Rightarrow \mathbf{CPV}\int_{\Gamma(\mathrm{x})} T_{ij}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x})u_k(\mathrm{Q})\,\mathrm{d}\Gamma(\mathrm{x}) - \frac{1}{2}u_i(\mathrm{Q}). \end{aligned} \tag{20}$$

It has to be noted that the term $x_k(\xi) - x_k(\mathrm{x})$ is of $\mathrm{O}(r)$ which will smooth the singular terms in the kernel T_{ij}

(which is of O($1/r$)) leading to zero jump term. Other integrals in Eq. (12) have weak singular kernels as integrands, which lead to smooth terms. Therefore, the final form of Eq. (12) can be written as follows:

$$\frac{1}{2}u_i(\xi) + \mathbf{CPV}\int_{\Gamma(x)} T_{ij}(\xi,x)(u_j(x) - D_{jk}(\xi,x)u_k(Q))\,d\Gamma(x) = \int_{\Gamma(x)} U_{ij}(\xi,x)t_j(x)\,d\Gamma(x) + \int_{\Gamma(x)}\left[V_{i,n}(\xi,x) - \frac{v}{(1-v)\lambda^2}U_{in}(\xi,x)\right]q\,d\Gamma(x) - \frac{1}{2}u_i(Q). \tag{21}$$

If the point Q is chosen to have the same place as that of ξ (i.e. $Q=\xi$), the singular terms in the integral in the LHS of Eq. (21) will vanish as x approaches ξ. Therefore, Eq. (21) can be re-written as follows:

$$u_i(\xi) + \int_{\Gamma(x)} T_{ij}(\xi,x)(u_j(x) - u_j(\xi))\,d\Gamma(x) = \int_{\Gamma(x)} U_{ij}(\xi,x)t_j(x)\,d\Gamma(x) + \int_{\Gamma(x)}\left[V_{i,n}(\xi,x) - \frac{v}{(1-v)\lambda^2}U_{in}(\xi,x)\right]q\,d\Gamma(x). \tag{22}$$

Eq. (22) can be used, without any special consideration, to compute boundary generalized displacements at any point ξ, as all involved integrals are either smooth or weak singular.

5. Stress resultant integral equations

In this section, the internal point integral equations for computing the stress-resultants (bending moments and shear forces) will be formed based on the proposed relative quantity formulation in Eqs. (12) and (22). Eq. (12) can be split into the following two integral equations (one for rotations and the other for the vertical deflection) as follows:

$$u_\alpha(\xi) + \int_{\Gamma(x)} T_{\alpha j}(\xi,x)(u_j(x) - D_{jk}(\xi,x)u_k(Q))\,d\Gamma(x) = \int_{\Gamma(x)} U_{\alpha j}(\xi,x)t_j(x)\,d\Gamma(x) + u_\alpha(Q) + \int_{\Gamma(x)}\left[V_{\alpha,n}(\xi,x) - \frac{v}{(1-v)\lambda^2}U_{\alpha n}(\xi,x)\right]q\,d\Gamma(x), \tag{23}$$

$$u_3(\xi) + \int_{\Gamma(x)} T_{3j}(\xi,x)(u_j(x) - D_{jk}(\xi,x)u_k(Q))\,d\Gamma(x) = \int_{\Gamma(x)} U_{3j}(\xi,x)t_j(x)\,d\Gamma(x) + u_3(Q) + \int_{\Gamma(x)}\left[V_{3,n}(\xi,x) - \frac{v}{(1-v)\lambda^2}U_{3n}(\xi,x)\right]q\,d\Gamma(x). \tag{24}$$

It has to be noted that choosing $u_k(Q)=0$ will make the proposed formulation in Eqs. (23) and (24) approaches the traditional BEM formulation in Refs. [2,3].

5.1. The bending moment integral equation

Differentiating Eq. (23) w.r.t the coordinates of the source point ($x_\beta(\xi)$), taking into account that the matrix D_{jk} is not constant (recall Eq. (19)), gives

$$u_{\alpha,\beta}(\xi) + \int_{\Gamma(x)} [T_{\alpha j,\beta}(\xi,x)(u_j(x) - D_{jk}(\xi,x)u_k(Q)) - T_{\alpha j}(\xi,x)D_{jk,\beta}(\xi,x)u_k(Q)]\,d\Gamma(x) = \int_{\Gamma(x)} U_{\alpha j,\beta}(\xi,x)t_j(x)\,d\Gamma(x) + \int_{\Gamma(x)}\left[V_{\alpha,\beta n}(\xi,x) - \frac{v}{(1-v)\lambda^2}U_{\alpha\theta,\beta}(\xi,x)n_\theta(x)\right] \times q\,d\Gamma(x), \tag{25}$$

where

$$D_{jk,\alpha}(\xi,x) = \frac{\partial D_{jk}(\xi,x)}{\partial x_\alpha(\xi)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta_{1\alpha} & \delta_{2\alpha} & 1 \end{bmatrix} = \delta_{k\alpha}\delta_{j3}. \tag{26}$$

Similarly, by changing indices one can easily obtain other relevant derivatives, such as $u_{\beta,\alpha}(\xi)$ and $u_{\gamma,\gamma}(\xi)$. It has to be noted that the following terms can be simplified as follows (recall Eq. (26)):

$$T_{\alpha j}(\xi,x)D_{jk,\beta}(\xi,x) = T_{\alpha 3}(\xi,x)\delta_{k\beta}, \tag{27}$$

and using suitable expansion, the following simplification can be also obtained:

$$T_{\gamma j}(\xi,x)D_{jk,\gamma}(\xi,x) = T_{\gamma 3}(\xi,x)D_{3k,\gamma}(\xi,x) = T_{k3}(\xi,x)(1-\delta_{k3}) \quad \text{with no summation on } k. \tag{28}$$

The following kernels can be defined:

$$U_{\alpha\beta j}(\xi,x) = \frac{D(1-v)}{2}\left(U_{\alpha j,\beta}(\xi,x) + U_{\beta j,\alpha}(\xi,x) + \frac{2v}{(1-v)}U_{\gamma j,\gamma}(\xi,x)\delta_{\alpha\beta}\right), \tag{29}$$

$$T_{\alpha\beta j}(\xi,x) = \frac{D(1-v)}{2}\left(T_{\alpha j,\beta}(\xi,x) + T_{\beta j,\alpha}(\xi,x) + \frac{2v}{(1-v)}T_{\gamma j,\gamma}(\xi,x)\delta_{\alpha\beta}\right), \tag{30}$$

$$W_{\alpha\beta}(\xi,x) = \frac{D(1-v)}{2}\left(V_{\alpha,\beta n}(\xi,x) + V_{\beta,\alpha n}(\xi,x) + \frac{2v}{(1-v)}V_{\gamma,\gamma n}(\xi,x)\delta_{\alpha\beta}\right) - \frac{v}{(1-v)\lambda^2}U_{\alpha\beta n}, \tag{31}$$

$$F_{\alpha\beta k}(\xi,\mathrm{x}) = \frac{D(1-v)}{2}(T_{\alpha 3}(\xi,\mathrm{x})\delta_{k\beta} + T_{\beta 3}(\xi,\mathrm{x})\delta_{k\alpha} + T_{k3}(\xi,\mathrm{x}) \times (1-\delta_{k3})) \quad \text{with no summation on } k. \tag{32}$$

Using suitable algebraic simplifications expression for the new kernel $F_{\alpha\beta k}(\xi,\mathrm{x})$ can be obtained as follows:

$$F_{\alpha\beta k}(\xi,\mathrm{x}) = \begin{bmatrix} DT_{13}(\xi,\mathrm{x}) & DvT_{23}(\xi,\mathrm{x}) & 0 \\ \frac{D(1-v)}{2}T_{23}(\xi,\mathrm{x}) & \frac{D(1-v)}{2}T_{13}(\xi,\mathrm{x}) & 0 \\ DvT_{13}(\xi,\mathrm{x}) & DT_{23}(\xi,\mathrm{x}) & 0 \end{bmatrix}. \tag{33}$$

Substituting Eq. (25) and similar derivatives into Eq. (1), the bending moment stress-resultant integral equation can be written as follows (after consideration of the definitions in Eqs. (27)–(32)):

$$\begin{aligned} M_{\alpha\beta}(\xi) = &\int_{\Gamma(\mathrm{x})} U_{\alpha\beta j}(\xi,\mathrm{x})t_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &- \int_{\Gamma(\mathrm{x})} T_{\alpha\beta j}(\xi,\mathrm{x})u_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \left[\int_{\Gamma(\mathrm{x})} [T_{\alpha\beta j}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x}) + F_{\alpha\beta k}(\xi,\mathrm{x})]\,\mathrm{d}\Gamma(\mathrm{x})\right]u_k(\mathrm{Q}) \\ &+ q\int_{\Gamma(\mathrm{x})} W_{\alpha\beta}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) + \frac{v}{(1-v)\lambda^2}q\delta_{\alpha\beta}. \end{aligned} \tag{34}$$

5.2. The shear force integral equation

Differentiating Eq. (24) w.r.t the coordinates of the source point ($x_\alpha(\xi)$), it gives

$$\begin{aligned} u_{3,\alpha}(\xi) + &\int_{\Gamma(\mathrm{x})} [T_{3j,\alpha}(\xi,\mathrm{x})(u_j(\mathrm{x}) - D_{jk}(\xi,\mathrm{x})u_k(\mathrm{Q})) \\ &- T_{3j}(\xi,\mathrm{x})D_{jk,\alpha}(\xi,\mathrm{x})u_k(\mathrm{Q})]\,\mathrm{d}\Gamma(\mathrm{x}) \\ &= \int_{\Gamma(\mathrm{x})} U_{3j,\alpha}(\xi,\mathrm{x})t_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \int_{\Gamma(\mathrm{x})} \left[V_{3,\alpha n}(\xi,\mathrm{x}) - \frac{v}{(1-v)\lambda^2}U_{3\theta,\alpha}(\xi,\mathrm{x})n_\theta(\mathrm{x})\right] \\ &\times q\,\mathrm{d}\Gamma(\mathrm{x}). \end{aligned} \tag{35}$$

Recalling Eq. (26), it is easy to show that

$$T_{3j}(\xi,\mathrm{x})D_{jk,\alpha}(\xi,\mathrm{x}) = T_{33}(\xi,\mathrm{x})\delta_{k\alpha}. \tag{36}$$

Correspondingly

$$T_{3j}(\xi,\mathrm{x})D_{jk,\alpha}(\xi,\mathrm{x})u_k(\mathrm{Q}) = T_{33}(\xi,\mathrm{x})u_\alpha(\mathrm{Q}). \tag{37}$$

So Eq. (35) can be re-written in the following form:

$$\begin{aligned} u_{3,\alpha}(\xi) + &\int_{\Gamma(\mathrm{x})} [T_{3j,\alpha}(\xi,\mathrm{x})(u_j(\mathrm{x}) - D_{jk}(\xi,\mathrm{x})u_k(\mathrm{Q})) \\ &- T_{33}(\xi,\mathrm{x})u_\alpha(\mathrm{Q})]\,\mathrm{d}\Gamma(\mathrm{x}) \\ &= \int_{\Gamma(\mathrm{x})} U_{3j,\alpha}(\xi,\mathrm{x})t_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \int_{\Gamma(\mathrm{x})} \left[V_{3,\alpha n}(\xi,\mathrm{x}) - \frac{v}{(1-v)\lambda^2}U_{3\theta,\alpha}(\xi,\mathrm{x})n_\theta(\mathrm{x})\right] \\ &\times q\,\mathrm{d}\Gamma(\mathrm{x}). \end{aligned} \tag{38}$$

Substituting from Eqs. (23) and (38) into Eq. (2), it gives

$$\begin{aligned} Q_{3\alpha}(\xi) &= \frac{D(1-v)}{2}\lambda^2\Bigg\{\int_{\Gamma(\mathrm{x})} U_{\alpha j}(\xi,\mathrm{x})t_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &- \int_{\Gamma(\mathrm{x})} T_{\alpha j}(\xi,\mathrm{x})u_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \left[\int_{\Gamma(\mathrm{x})} T_{\alpha j}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x})\right] + u_\alpha(\mathrm{Q}) \\ &+ \int_{\Gamma(\mathrm{x})} U_{3j,\alpha}(\xi,\mathrm{x})t_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &- \int_{\Gamma(\mathrm{x})} T_{3j,\alpha}(\xi,\mathrm{x})u_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \left[\int_{\Gamma(\mathrm{x})} [T_{3j,\alpha}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x}) + T_{33}(\xi,\mathrm{x})\delta_{k\alpha}]\,\mathrm{d}\Gamma(\mathrm{x})\right]u_k(\mathrm{Q}) \\ &+ \int_{\Gamma(\mathrm{x})} \left[V_{\alpha,n}(\xi,\mathrm{x}) - \frac{v}{(1-v)\lambda^2}U_{\alpha n}(\xi,\mathrm{x})\right]q\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \int_{\Gamma(\mathrm{x})} \left[V_{3,\alpha n}(\xi,\mathrm{x}) - \frac{v}{(1-v)\lambda^2}U_{3\theta,\alpha}(\xi,\mathrm{x})n_\theta(\mathrm{x})\right] \\ &\times q\,\mathrm{d}\Gamma(\mathrm{x})\Bigg\}. \end{aligned} \tag{39}$$

The following kernels can be defined:

$$U_{3\alpha j}(\xi,\mathrm{x}) = \frac{D(1-v)}{2}\lambda^2(U_{\alpha j}(\xi,\mathrm{x}) + U_{3j,\alpha}(\xi,\mathrm{x})), \tag{40}$$

$$T_{3\alpha j}(\xi,\mathrm{x}) = \frac{D(1-v)}{2}\lambda^2(T_{\alpha j}(\xi,\mathrm{x}) + T_{3j,\alpha}(\xi,\mathrm{x})), \tag{41}$$

$$W_{3\beta}(\xi,\mathrm{x}) = \frac{D(1-v)}{2}\lambda^2(V_{\beta,n}(\xi,\mathrm{x}) + V_{3,\beta n}(\xi,\mathrm{x})) - \frac{v}{(1-v)\lambda^2}U_{3\beta n}. \tag{42}$$

After consideration the definitions in Eqs. (40)–(42), Eq. (39) can be re-written as follows:

$$\begin{aligned} Q_{3\alpha}(\xi) = &\int_{\Gamma(\mathrm{x})} U_{3\alpha j}(\xi,\mathrm{x})t_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &- \int_{\Gamma(\mathrm{x})} T_{3\alpha j}(\xi,\mathrm{x})u_j(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\ &+ \left[\int_{\Gamma(\mathrm{x})} T_{3\alpha j}(\xi,\mathrm{x})D_{jk}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x})\right]u_k(\mathrm{Q}) \\ &+ \frac{D(1-v)}{2}\lambda^2\left[\int_{\Gamma(\mathrm{x})} T_{33}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) + 1\right]u_\alpha(\mathrm{Q}) \\ &+ q\int_{\Gamma(\mathrm{x})} W_{3\alpha}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}). \end{aligned} \tag{43}$$

Both Eqs. (34) and (43) are the relative quantity integral equation for the bending moment and the shear stress resultants, respectively, at internal point ξ. It has to be noted that choosing $u_k(\mathrm{Q}) = 0$ will make the proposed formulation in Eqs. (34) and (43) approaches the traditional BEM formulation in Refs. [2,3].

6. Boundary stress resultant integral equations

In this section, Eqs. (34) and (43) will be discussed as the point ξ will move to smooth boundary.

6.1. Bending moment boundary integral equation

Eq. (34) can be expanded in the following form:

$$M_{\alpha\beta}(\xi) = \int_{\Gamma(x)} U_{\alpha\beta\gamma}(\xi, x)t_\gamma(x)\,d\Gamma(x) + \int_{\Gamma(x)} U_{\alpha\beta3}(\xi, x)t_3(x)\,d\Gamma(x) - \int_{\Gamma(x)} T_{\alpha\beta\gamma}(\xi, x) \times [u_\gamma(x) - D_{\gamma k}(\xi, x)u_k(Q)]\,d\Gamma(x) - \int_{\Gamma(x)} T_{\alpha\beta3}(\xi, x) \times [u_3(x) - D_{3k}(\xi, x)u_k(Q)]\,d\Gamma(x) + \left[\int_{\Gamma(x)} F_{\alpha\beta k}(\xi, x)\,d\Gamma(x)\right]u_k(Q) + q\int_{\Gamma(x)} W_{\alpha\beta}(\xi, x)\,d\Gamma(x) + \frac{v}{(1-v)\lambda^2}q\delta_{\alpha\beta}, \tag{44}$$

which can be symbolized in the following form:

$$M_{\alpha\beta}(\xi) = I_1 + I_2 - I_3 - I_4 + I_5 + I_6, \tag{45}$$

where each symbol from Eq. (45) has to match the corresponding integral in Eq. (44). Now each integral in Eq. (44) will be considered as ξ will move to be a boundary point. Consider Fig. 1 and the relationships in Eqs. (13)–(17) together with suitable Taylor expansion [4], the following jump terms can be computed. For the sake of shortening the present paper, some of these integrals will have the same jump term as those obtained by the author and his co-workers in Ref. [4], therefore herein the final results will be given (the interested reader can refer to Ref. [4] for detailed discussion). In the paragraphs below, only the new forms of integrals will be considered in details. Now each integral will be considered:

$$I_1 \Rightarrow \textbf{CPV}\int_{\Gamma(x)} U_{\alpha\beta\gamma}(\xi, x)t_\gamma(x)\,d\Gamma(x) + \frac{3v-1}{16}M_{\gamma\gamma}(\xi)\delta_{\alpha\beta} - \frac{2(v-3)}{16}M_{\alpha\beta}(\xi), \tag{46}$$

$$I_2 \Rightarrow \textbf{CPV}\int_{\Gamma(x)} U_{\alpha\beta3}(\xi, x)t_3(x)\,d\Gamma(x). \tag{47}$$

Using suitable Taylor expansion the final jump term of the integral I_3 can be obtained as follows (noting that $u_{\alpha,\beta}(Q) = 0$ and $D_{\gamma3} = 0$):

$$I_3 \Rightarrow \textbf{HFP}\int_{\Gamma(x)} T_{\alpha\beta\gamma}(\xi, x)[u_\gamma(x) - D_{\gamma k}(\xi, x)u_k(Q)]\,d\Gamma(x) - \frac{D(1+v)(1-v)}{16}[u_{\beta,\alpha}(\xi) + u_{\alpha,\beta}(\xi) + \delta_{\alpha\beta}u_{\gamma,\gamma}(\xi)], \tag{48}$$

where the symbol **HFP** denotes the following integral will be interpreted in Hadamard finite part sense.

$$I_4 \Rightarrow \textbf{CPV}\int_{\Gamma(x)} T_{\alpha\beta3}(\xi, x)[u_3(x) - D_{3k}(\xi, x)u_k(Q)]\,d\Gamma(x), \tag{49}$$

$$I_5 \Rightarrow \left[\int_{\Gamma(x)} F_{\alpha\beta k}(\xi, x)\,d\Gamma(x)\right]u_k(Q), \tag{50}$$

$$I_6 \Rightarrow q \times \textbf{CPV}\int_{\Gamma(x)} W_{\alpha\beta}(\xi, x)\,d\Gamma(x) - \frac{vq}{(1-v)\lambda^2}\frac{1+v}{4}\delta_{\alpha\beta}. \tag{51}$$

Substituting from Eqs. (46)–(51) and simplifying taking into account the following relationship (recall Eq. (1)):

$$M_{\gamma\gamma}(\xi) = D\frac{1+v}{1-v}u_{\gamma,\gamma} + \frac{2vq}{(1-v)\lambda^2}. \tag{52}$$

The final boundary integral equation is

$$\frac{1}{2}M_{\alpha\beta}(\xi) = \textbf{CPV}\int_{\Gamma(x)} U_{\alpha\beta\gamma}(\xi, x)t_\gamma(x)\,d\Gamma(x) + \textbf{CPV}\int_{\Gamma(x)} U_{\alpha\beta3}(\xi, x)t_3(x)\,d\Gamma(x) - \textbf{HFP}\int_{\Gamma(x)} T_{\alpha\beta\gamma}(\xi, x) \times [u_\gamma(x) - D_{\gamma k}(\xi, x)u_k(Q)]\,d\Gamma(x) - \textbf{CPV}\int_{\Gamma(x)} T_{\alpha\beta3}(\xi, x) \times [u_3(x) - D_{3k}(\xi, x)u_k(Q)]\,d\Gamma(x) + \left[\int_{\Gamma(x)} F_{\alpha\beta k}(\xi, x)\,d\Gamma(x)\right]u_k(Q) + q \times \textbf{CPV}\int_{\Gamma(x)} W_{\alpha\beta}(\xi, x)\,d\Gamma(x) + \frac{v}{(1-v)\lambda^2}q\delta_{\alpha\beta}. \tag{53}$$

The integral equation in Eq. (53) can be used to compute the bending moment stress resultants at boundary point ξ. If the point ξ is chosen to be Q, Eq. (53) can be re-written as

$$\frac{1}{2}M_{\alpha\beta}(\xi) = \textbf{CPV}\int_{\Gamma(x)} U_{\alpha\beta\gamma}(\xi, x)t_\gamma(x)\,d\Gamma(x) + \textbf{CPV}\int_{\Gamma(x)} U_{\alpha\beta3}(\xi, x)t_3(x)\,d\Gamma(x) - \int_{\Gamma(x)} T_{\alpha\beta\gamma}(\xi, x) \times [u_\gamma(x) - D_{\gamma k}(\xi, x)u_k(\xi)]\,d\Gamma(x)$$

$$- \int_{\Gamma(x)} T_{\alpha\beta 3}(\xi, x) \times [u_3(x) - D_{3k}(\xi, x) u_k(\xi)]\, d\Gamma(x) + \left[\int_{\Gamma(x)} F_{\alpha\beta k}(\xi, x)\, d\Gamma(x)\right] u_k(\xi) + q \times \mathbf{CPV} \int_{\Gamma(x)} W_{\alpha\beta}(\xi, x)\, d\Gamma(x) + \frac{v}{(1-v)\lambda^2} q\delta_{\alpha\beta}. \quad (54)$$

It can be seen that all HFP integrals vanish. Therefore, this integral equation can be used on the boundary without any special considerations.

6.2. Shear force boundary integral equation

Eq. (43) can be expanded in the following form:

$$Q_{3\alpha}(\xi) = \int_{\Gamma(x)} U_{3\alpha\gamma}(\xi, x) t_\gamma(x)\, d\Gamma(x) + \int_{\Gamma(x)} U_{3\alpha 3}(\xi, x) t_3(x)\, d\Gamma(x) - \int_{\Gamma(x)} T_{3\alpha\gamma}(\xi, x) \times [u_\gamma(x) - D_{\gamma k}(\xi, x) u_k(Q)]\, d\Gamma(x) - \int_{\Gamma(x)} T_{3\alpha 3}(\xi, x) \times [u_3(x) - D_{3k}(\xi, x) u_k(Q)]\, d\Gamma(x) + \frac{D(1-v)}{2}\lambda^2 \left[\int_{\Gamma(x)} T_{33}(\xi, x)\, d\Gamma(x)\right] u_\alpha(Q) + \frac{D(1-v)}{2}\lambda^2 u_\alpha(Q) + q \int_{\Gamma(x)} W_{3\alpha}(\xi, x)\, d\Gamma(x) \quad (55)$$

or it can be re-written as

$$Q_{3\alpha}(\xi) = I_7 + I_8 - I_9 - I_{10} + \frac{D(1-v)}{2}\lambda^2 I_{11} u_\alpha(Q) + \frac{D(1-v)}{2}\lambda^2 u_\alpha(Q) + I_{12}. \quad (56)$$

Similar to the boundary integral equation for bending moment each of the integrals in Eqs. (55) and (56) will be considered as the point ξ will move to smooth boundary. Consider Fig. 1 and the relationships in Eqs. (13)–(17) together with suitable Taylor expansion [4], the following jump terms can be computed:

$$I_7 \Rightarrow \int_{\Gamma(x)} U_{3\alpha\gamma}(\xi, x) t_\gamma(x)\, d\Gamma(x), \quad (57)$$

$$I_8 \Rightarrow \mathbf{CPV} \int_{\Gamma(x)} U_{3\alpha 3}(\xi, x) t_3(x)\, d\Gamma(x) + \frac{Q_{3\alpha}(\xi)}{4}, \quad (58)$$

$$I_9 \Rightarrow \mathbf{CPV} \int_{\Gamma(x)} T_{3\alpha\gamma}(\xi, x) \times [u_\gamma(x) - D_{\gamma k}(\xi, x) u_k(Q)]\, d\Gamma(x) - \frac{D(1-v)\lambda^2}{8}[u_\alpha(\xi) - u_\alpha(Q)]. \quad (59)$$

The integral I_{10} will be considered in more detail as it encounters new developments:

$$I_{10} = \int_{\Gamma(x)} T_{3\alpha 3}(\xi, x)[u_3(x) - D_{3k}(\xi, x) u_k(Q)]\, d\Gamma(x) = \int_{\Gamma(x)} T_{3\alpha 3}(\xi, x)[u_3(x) - D_{3\theta}(\xi, x) u_\theta(Q)]\, d\Gamma(x) + \int_{\Gamma(x)} T_{3\alpha 3}(\xi, x)[u_3(x) - D_{31}(\xi, x) u_3(Q)]\, d\Gamma(x) \quad (60)$$

$$= I_{101} + I_{102}. \quad (61)$$

Each of the integrals I_{101} and I_{102} will be considered separately. Note that

$$D_{33}(\xi, x) = 1. \quad (62)$$

The integral I_{101} will be considered in HFP sense as the kernel $T_{3\alpha 3}$ is hyper-singular of $O(1/r^2)$:

$$I_{101} \Rightarrow \mathbf{HFP} \int_{\Gamma(x)} T_{3\alpha 3}(\xi, x)[u_3(x) - D_{3\theta}(\xi, x) u_\theta(Q)]\, d\Gamma(x) - \frac{D(1-v)\lambda^2}{8} u_{3,\alpha}(\xi). \quad (63)$$

It has to be noted that in Eq. (63) the derivative $u_{3,\alpha}(Q)$ is considered to be zero. From Eq. (19), it can be seen that

$$D_{3\theta}(\xi, x) = x_\theta(\xi) - x_\theta(x) \quad (64)$$

which is of $O(r)$, therefore the term $D_{3\theta}$ in the integral I_{102} will smooth the kernel $T_{3\alpha 3}$ by one order, then the product $D_{3\theta}\ T_{3\alpha 3}$ will be of $O(1/r)$. Consequently the integral I_{102} will be considered in Cauchy principal value sense as follows:

$$I_{102} \Rightarrow \mathbf{CPV} \int_{\Gamma(x)} T_{3\alpha 3}(\xi, x)[u_3(x) - D_{31}(\xi, x) u_3(Q)]\, d\Gamma(x) - \frac{D(1-v)\lambda^2}{8}[u_3(\xi) - u_3(Q)]. \quad (65)$$

The integral I_{11} will lead to the well-known $\frac{1}{2}$ jump term:

$$I_{11} \Rightarrow \mathbf{CPV} \int_{\Gamma(x)} T_{33}(\xi, x)\, d\Gamma(x) + \left(-\frac{1}{2}\right). \quad (66)$$

The integral I_{12} contains weak singular kernel so it will not led to any jump term:

$$I_{12} \Rightarrow q \int_{\Gamma(x)} W_{3\alpha}(\xi, x)\, d\Gamma(x). \quad (67)$$

Substituting from Eqs. (57)–(67) into (55) and simplifying, it gives

$$
\begin{aligned}
\frac{1}{2}Q_{3\alpha}(\xi) = {} & \int_{\Gamma(\mathrm{x})} U_{3\alpha\gamma}(\xi,\mathrm{x})t_\gamma(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\
& + \mathbf{CPV}\int_{\Gamma(\mathrm{x})} U_{3\alpha 3}(\xi,\mathrm{x})t_3(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\
& - \mathbf{CPV}\int_{\Gamma(\mathrm{x})} T_{3\alpha\gamma}(\xi,\mathrm{x})[u_\gamma(\mathrm{x}) - D_{\gamma k}(\xi,\mathrm{x})u_k(\mathrm{Q})]\,\mathrm{d}\Gamma(\mathrm{x}) \\
& - \mathbf{HFP}\int_{\Gamma(\mathrm{x})} T_{3\alpha 3}(\xi,\mathrm{x})[u_3(x) - D_{3\theta}(\xi,\mathrm{x})u_\theta(\mathrm{Q})]\,\mathrm{d}\Gamma(\mathrm{x}) \\
& - \mathbf{CPV}\int_{\Gamma(\mathrm{x})} T_{3\alpha 3}(\xi,\mathrm{x})[u_3(\mathrm{x}) - D_{31}(\xi,\mathrm{x})u_3(\mathrm{Q})]\,\mathrm{d}\Gamma(\mathrm{x}) \\
& + \frac{D(1-\upsilon)}{2}\lambda^2\left[\mathbf{CPV}\int_{\Gamma(\mathrm{x})} T_{33}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x})\right]u_\alpha(\mathrm{Q}) \\
& + \frac{D(1-\upsilon)}{4}\lambda^2 u_\alpha(\mathrm{Q}) \\
& + q\int_{\Gamma(\mathrm{x})} W_{3\alpha}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}). \qquad (68)
\end{aligned}
$$

The integral equation in Eq. (68) can be used to compute the shear force stress resultants at boundary point ξ. If the point ξ is chosen to be Q, Eq. (68) can be re-written as

$$
\begin{aligned}
\frac{1}{2}Q_{3\alpha}(\xi) = {} & \int_{\Gamma(\mathrm{x})} U_{3\alpha\gamma}(\xi,\mathrm{x})t_\gamma(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\
& + \mathbf{CPV}\int_{\Gamma(\mathrm{x})} U_{3\alpha 3}(\xi,\mathrm{x})t_3(\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}) \\
& - \int_{\Gamma(\mathrm{x})} T_{3\alpha\gamma}(\xi,\mathrm{x}) \\
& \times [u_\gamma(\mathrm{x}) - D_{\gamma k}(\xi,\mathrm{x})u_k(\xi)]\,\mathrm{d}\Gamma(\mathrm{x}) \\
& - \int_{\Gamma(\mathrm{x})} T_{3\alpha 3}(\xi,\mathrm{x}) \\
& \times [u_3(\mathrm{x}) - D_{3k}(\xi,\mathrm{x})u_k(\xi)]\,\mathrm{d}\Gamma(\mathrm{x}) \\
& + \frac{D(1-\upsilon)}{2}\lambda^2\left[\mathbf{CPV}\int_{\Gamma(\mathrm{x})} T_{33}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x})\right]u_\alpha(\xi) \\
& + \frac{D(1-\upsilon)}{4}\lambda^2 u_\alpha(\xi) \\
& + q\int_{\Gamma(\mathrm{x})} W_{3\alpha}(\xi,\mathrm{x})\,\mathrm{d}\Gamma(\mathrm{x}). \qquad (69)
\end{aligned}
$$

It can be seen that all HFP integrals vanish. Therefore, this integral equation can be used on the boundary without any special considerations.

7. Numerical solution

Eqs. (54) and (69) are implemented into computer program which uses curved quadratic continuous elements. The weak singularity is treated using non-linear coordinate transformation [14] and the CPV integrals are computed using the proposed scheme in Ref. [9].

It has to be noted that the present formulation will be used to compute values at the boundary. However, the traditional BEM formulation [2,3] can be used to compute values at internal points which were extensively studied previously by the author in Refs. [2,3].

8. Example problem

In this example the present formulation is used to compute the boundary stress resultants for the shown semicircular cantilever slab in Fig. 2. The same problem was previously considered in Ref. [5] where the boundary stress resultants were computed using the stress resultant integral equations technique (SIE) and the shape function derivatives technique (SFD). The slab thickness is 0.3 m, modulus of elasticity is 2100 t/m^2, Poisson's ratio is 0.2. The slab is analysed under domain loading equal to -0.2 t/m^2. In Ref. [5] 10 fully discontinuous boundary elements were used to discretize the clamped part of the boundary, whereas 20 discontinuous elements were used to discretize the free edge part. Herein, the same discretization were employed but with fully continuous elements. Discontinuous elements are used at corners.

Figs. 3–7 demonstrate the stress resultants (moments and shear forces) along the circumference coordinate x (see Fig. 2). It can be seen that:

(1) The SFD technique cannot compute the tangential component for the bending moment along the clamed edge (as all generalized displacements are zeros).
(2) The present formulation results using the traditional continuous quadratic elements are very accurate compared to the SIE technique.

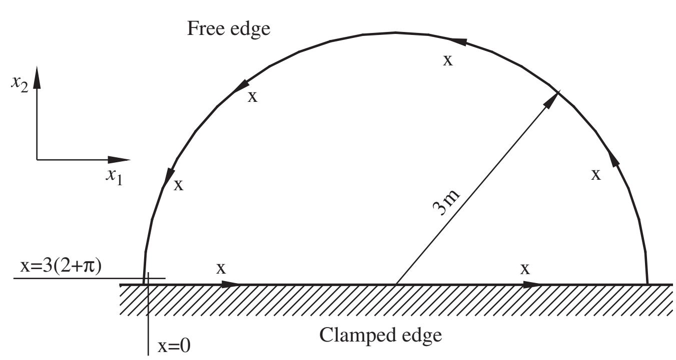

Fig. 2. The considered example problem.

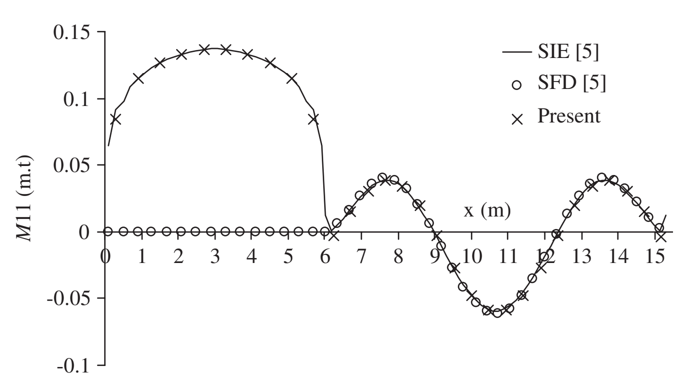

Fig. 3. The boundary bending moment M_{11}.

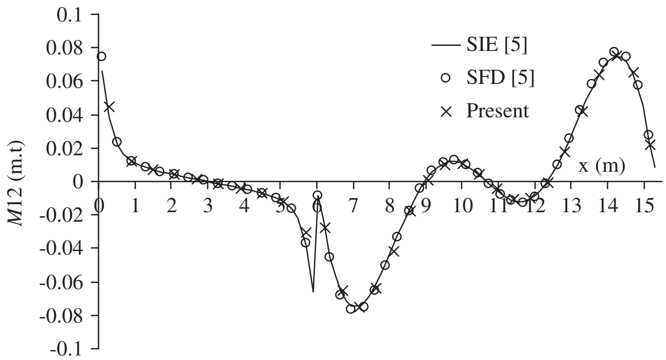

Fig. 4. The boundary twisting moment M_{12}.

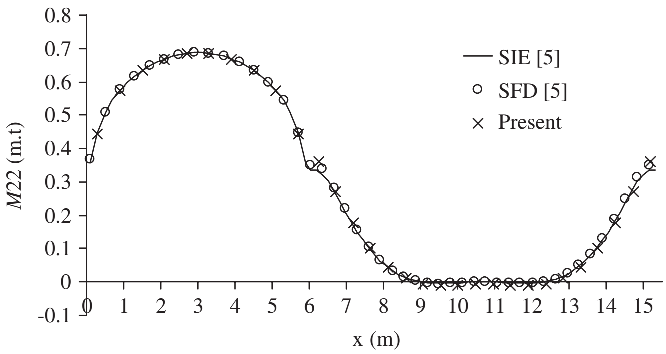

Fig. 5. The boundary bending moment M_{22}.

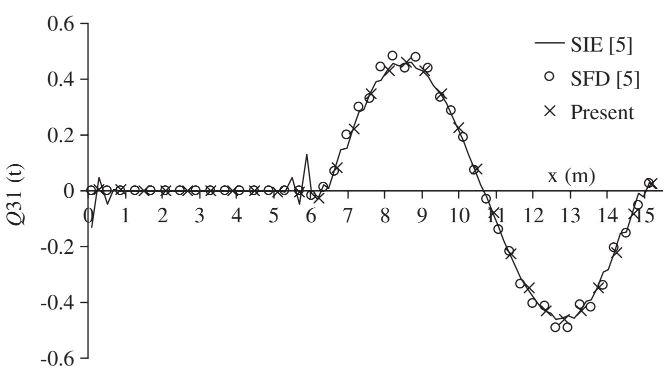

Fig. 6. The boundary shear force Q_{31}.

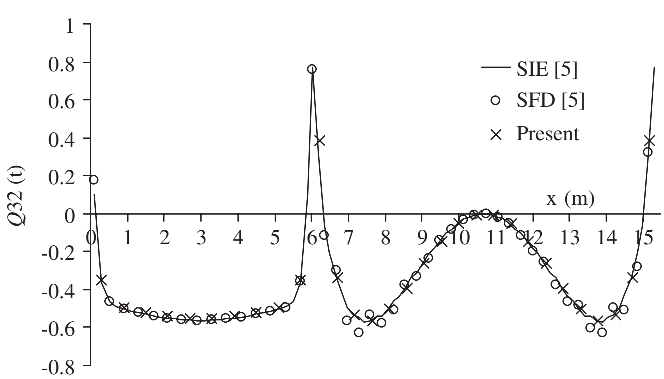

Fig. 7. The boundary shear force Q_{32}.

(3) There are some oscillations in the results for the shear along the free edge in the SFD technique, whereas the obtained results from the present formulation are accurate.
(4) There are some oscillations in the SIE results near the corners, whereas the present formulation results are smooth.

Generally it can be seen that the present formulation results are accurate and smooth regardless of the type of the boundary condition. It can be also seen that the traditional continuous quadratic boundary element is used without any special consideration.

9. Conclusions

The present paper demonstrated new integral equation formulation for shear deformable plate bending problems based on the relative displacement quantities. Such equations together with the traditional integral equations can be used easily to compute values anywhere in the plate problem, i.e. inside the domain (using the traditional integral equations [2,3]) or on the boundary (using the proposed formulation). The traditional continuous quadratic element can be used without computation of hyper singular integrals. An example problem of curved slab was presented to demonstrate the accuracy of the present formulation. Corner points will be considered in future study.

References

[1] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques: theory and applications in engineeting. Berlin, Heidelberg: Springer; 1984.
[2] Rashed YF. A boundary/domain element method for analysis of building raft foundations. Eng Anal Boundary Elem 2005;29: 859–77.
[3] Rashed YF. Boundary element modeling of flat plate floors under vertical loading. Int J Numer Methods Eng 2005;62: 1606–35.
[4] Rashed YF, Aliabadi MH, Brebbia CA. Hypersingular boundary element formulation for Reissner plates. Int J Solids Struct 1998;35(18):2229–49.
[5] Rashed YF, Aliabadi MH, Brebbia CA. On the evaluation of the stresses in the BEM for Reissner plate bending problems. Appl Math Modelling 1997;21:155–63.
[6] Stern M. A general boundary integral formulation for the numerical solution of plate bending problems. Int J Solids Struct 1979;15: 769–82.
[7] Bèzine G. A boundary integral equation method for plate flexure with conditions inside the domain. Int J Numer Methods Eng 1981;15:1647–57.
[8] Vander Weeën F. Application of the boundary integral equation method to Reissner's plate model. Int J Numer Methods Eng 1982;18: 1–10.
[9] Aliabadi MH, Rooke DP. Numerical fracture mechanics. Dordrecht: Kluwer Academic Publishers; 1991.
[10] Rashed YF. Boundary element formulations for thick plates. In: Topics in engineering, vol. 35. Southampton and Boston: WIT press; 2000.

[11] Kawahara T, Kisu H. Boundary element analysis system based on a formulation with relative quantity. In: Brebbia CA, editor. Boundary Elements X. Berlin: Springer; 1988. p. 111–21.

[12] Li Y-S, Obata T. Some improvements of accuracy and efficiency in three dimentional direct boundary element method. Int J Numer Methods Eng 1992;33:1451–64.

[13] Reissner E. On bending of elastic plates. Q Appl Math 1947; 5:55–68.

[14] Telles JCF. A self-adaptive coordinate transformation for efficient numerical evaluation of general boundary element integrals. Int J Numer Methods Eng 1987;24:959–73.