بسم الله الرحمن الرحيم

3D CFD Steady and Unsteady

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Contents

Importance of CFD

CFD is an important subject in computational mechanics as the application of fluid dynamics are so many is our life such as:

Airplane after-flow

Flow around buildings

Applications in Biomechanics
Applications in climate problems

Navier Stokes equations

Navier Stokes equations for viscous incompressible flow:

Equation of mass conservation (continuity equ.)

$$\dot{\boldsymbol{u}}_{i,i}(\mathrm{x}) = 0 \qquad\qquad i = 1 \rightarrow 3$$

Equation of momentum (dimensionless)

$$-p_{,i}(\mathrm{x}) + \frac{1}{Re}\left(\dot{\boldsymbol{u}}_{i,jj}(\mathrm{x}) + \dot{\boldsymbol{u}}_{j,ij}(\mathrm{x})\right) = \dot{\boldsymbol{u}}_j(\mathrm{x})\dot{\boldsymbol{u}}_{i,j}(\mathrm{x}) + \frac{\partial \dot{\boldsymbol{u}}_i}{\partial t}(\mathrm{x})$$

Pressure gradient

Convective term

Inertia term

Penalty formulations

Eliminate the pressure term assuming $p(\mathrm{x}) = -\lambda \dot{\boldsymbol{u}}_{i,i}(\mathrm{x})$

λ **must be large enough**

Rearranging Navier Stokes equations as follows:

$$\left(\boldsymbol{\lambda} + \frac{1}{Re}\right)\dot{\boldsymbol{u}}_{j,ij}(\mathrm{x}) + \frac{1}{Re}\dot{\boldsymbol{u}}_{i,jj}(\mathrm{x}) = \dot{\boldsymbol{u}}_j(\mathrm{x})\dot{\boldsymbol{u}}_{i,j}(\mathrm{x}) + \frac{\partial \dot{\boldsymbol{u}}_i}{\partial t}(\mathrm{x})$$

Comparing to Navier equation for elasticity:

$$(\lambda + \mu)u_{j,ij}(\mathrm{x}) + \mu u_{i,jj}(\mathrm{x}) = -b_i(\mathrm{x}) + \frac{\partial u_i}{\partial t}(\mathrm{x})$$

domain term

Inertia term

The analogy between N-S equations and N equations will be as follows:

$$\dot{\boldsymbol{u}}_i(\mathrm{x}) \rightarrow u_i(\mathrm{x})$$

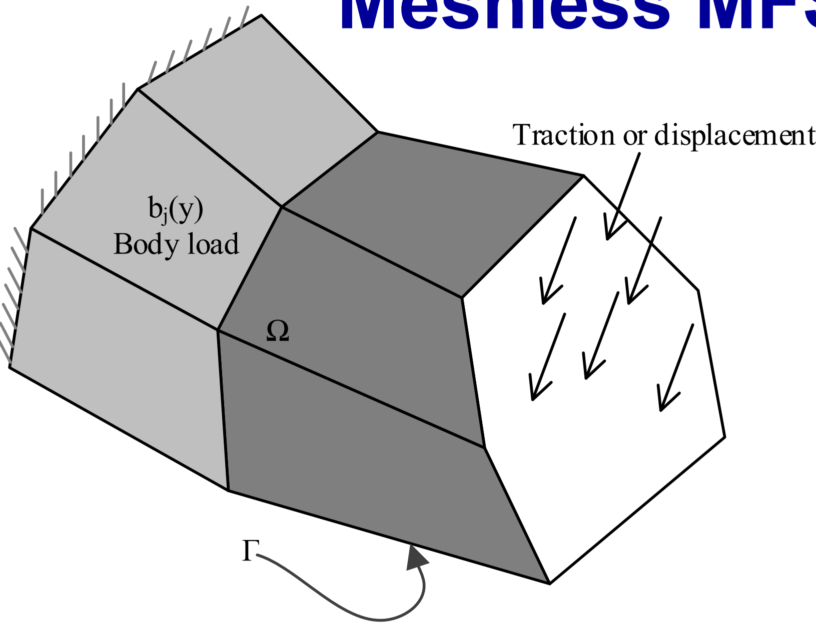
$$\dot{\boldsymbol{u}}_j(\mathrm{x})\dot{\boldsymbol{u}}_{i,j}(\mathrm{x}) \rightarrow -b_i(\mathrm{x})$$

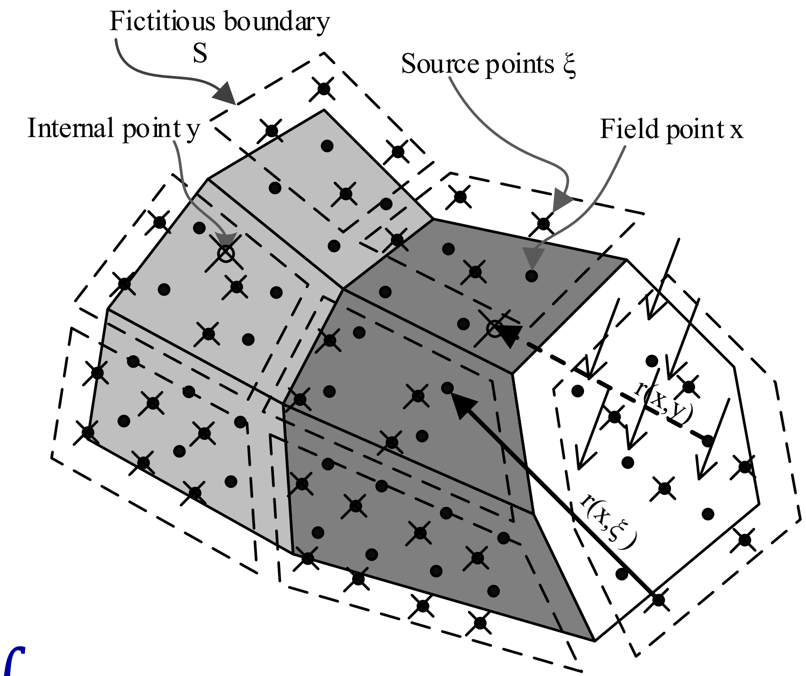
$$\frac{1}{Re} \rightarrow \mu$$

$$\boldsymbol{\lambda} \rightarrow \lambda$$

Steady state CFD for 3D problems

Meshless MFS for Single domain





Equation for displacement:

$$u_j(\mathrm{x}) = \sum_{k=1}^{k=\mathrm{N}} U_{ij}^*(\xi_k, \mathrm{x})\phi_i(\xi_k) + \int_{\Omega(\mathrm{y})} U_{ij}^*(\mathrm{y}, \mathrm{x}) b_i(y) d\Omega(\mathrm{y})$$

Equation for traction:

$$t_j(\mathrm{x}) = \sum_{k=1}^{k=\mathrm{N}} T_{ij}^*(\xi_k, \mathrm{x})\phi_i(\xi_k) + \int_{\Omega(\mathrm{y})} T_{ij}^*(\mathrm{y}, \mathrm{x}) b_i(y) d\Omega(\mathrm{y})$$

The matrix form:

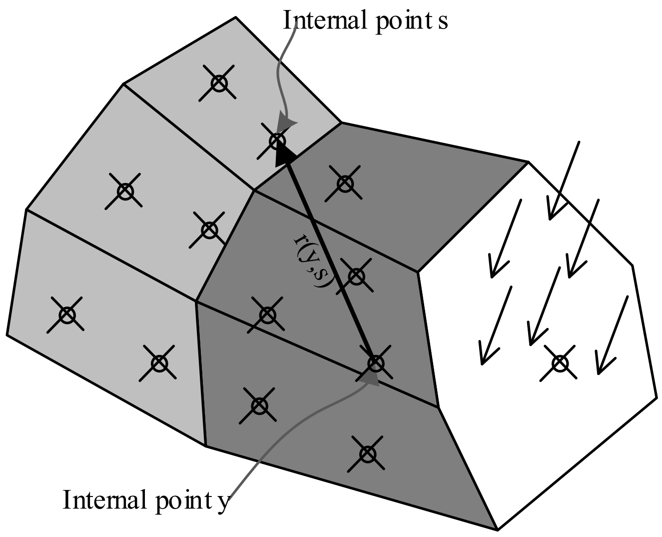
$$\begin{Bmatrix} \{\bar{u}(\mathrm{x})\}_{\mathbf{3}\mathrm{N}_1\times 1} \\ \{\bar{t}(\mathrm{x})\}_{\mathbf{3}\mathrm{N}_2\times 1} \end{Bmatrix} = \begin{bmatrix} [U^*(\xi,\mathrm{x})]_{\mathbf{3}\mathrm{N}_1\times\mathbf{3}\mathrm{N}} \\ [T^*(\xi,\mathrm{x})]_{\mathbf{3}\mathrm{N}_2\times\mathbf{3}\mathrm{N}} \end{bmatrix} \quad \{\phi(\xi)\}_{\mathbf{3}\mathrm{N}\times 1} + \left\{b_p(\mathrm{x})\right\}_{\mathbf{3}\mathrm{N}\times 1}$$

The convective term calculation:

$$u_{j,m}(\mathrm{s}) = \sum_{k=1}^{k=\mathrm{M}} U^*_{ij,m}(\xi_k,\mathrm{s})\phi_{\boldsymbol{i}}(\xi_k) + \int_{\Omega(\mathrm{y})} U^*_{ij,m}(\mathrm{y},\mathrm{s})b_i(y)d\Omega(\mathrm{y})$$

Treatment of domain integral

Using Monte-Carlo integration technique



$$\int_{\Omega(\mathrm{y})} U_{ij}^{*}(\mathrm{y},\mathrm{x}) b_i(y) d\Omega(\mathrm{y}) = \frac{\Omega}{\mathrm{N}_m} \sum_{m=1}^{m=\mathrm{N}_m} U_{ij}^{*}(\mathrm{y}_m,\mathrm{x}) b_i(y)$$

Solution procedure:

- **Initially compute the matrices .**
- **Start the loop on the nonlinear term convective terms. compute domain term .**
- **Solve the system of equations and get ϕ the get then boundary velocity and traction.**

$$\begin{Bmatrix} \{\bar{u}(\mathrm{x})\}_{3\mathrm{N}_1\times 1} \\ \{\bar{t}(\mathrm{x})\}_{3\mathrm{N}_2\times 1} \end{Bmatrix} = \begin{bmatrix} [U^*(\xi,\mathrm{x})]_{3\mathrm{N}_1\times 3\mathrm{N}} \\ [T^*(\xi,\mathrm{x})]_{3\mathrm{N}_2\times 3\mathrm{N}} \end{bmatrix} \{\phi(\xi)\}_{3\mathrm{N}\times 1} + \{b_p(\mathrm{x})\}_{3\mathrm{N}\times 1}$$

- **Compute the internal velocity and its derivative.**
- **Compute the convective term:**

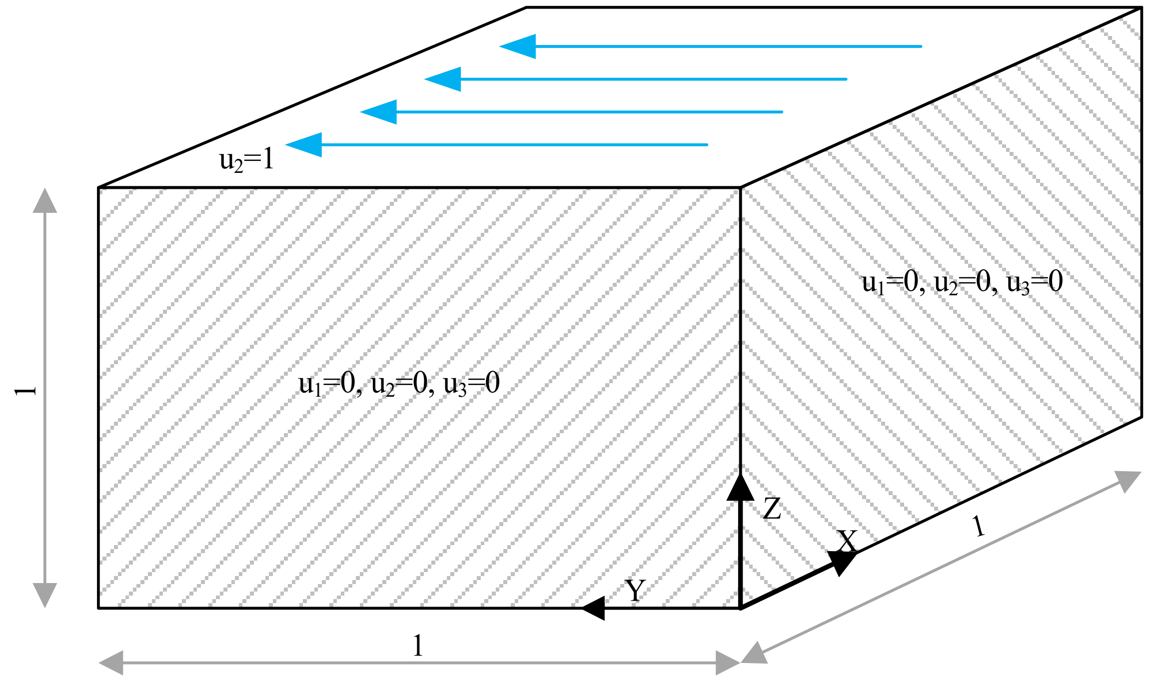
$$b_i(y) = \mathrm{K}\left(b_i(y)\right)^{(l)} + (\mathrm{K}-1)\left(b_i(y)\right)^{(l-1)}$$

- **Compute the difference in results of internal velocities.**

$$diff = \frac{\left(u_j^{(l)} - u_j^{(l-1)}\right)100}{u_j^{(l-1)}}$$

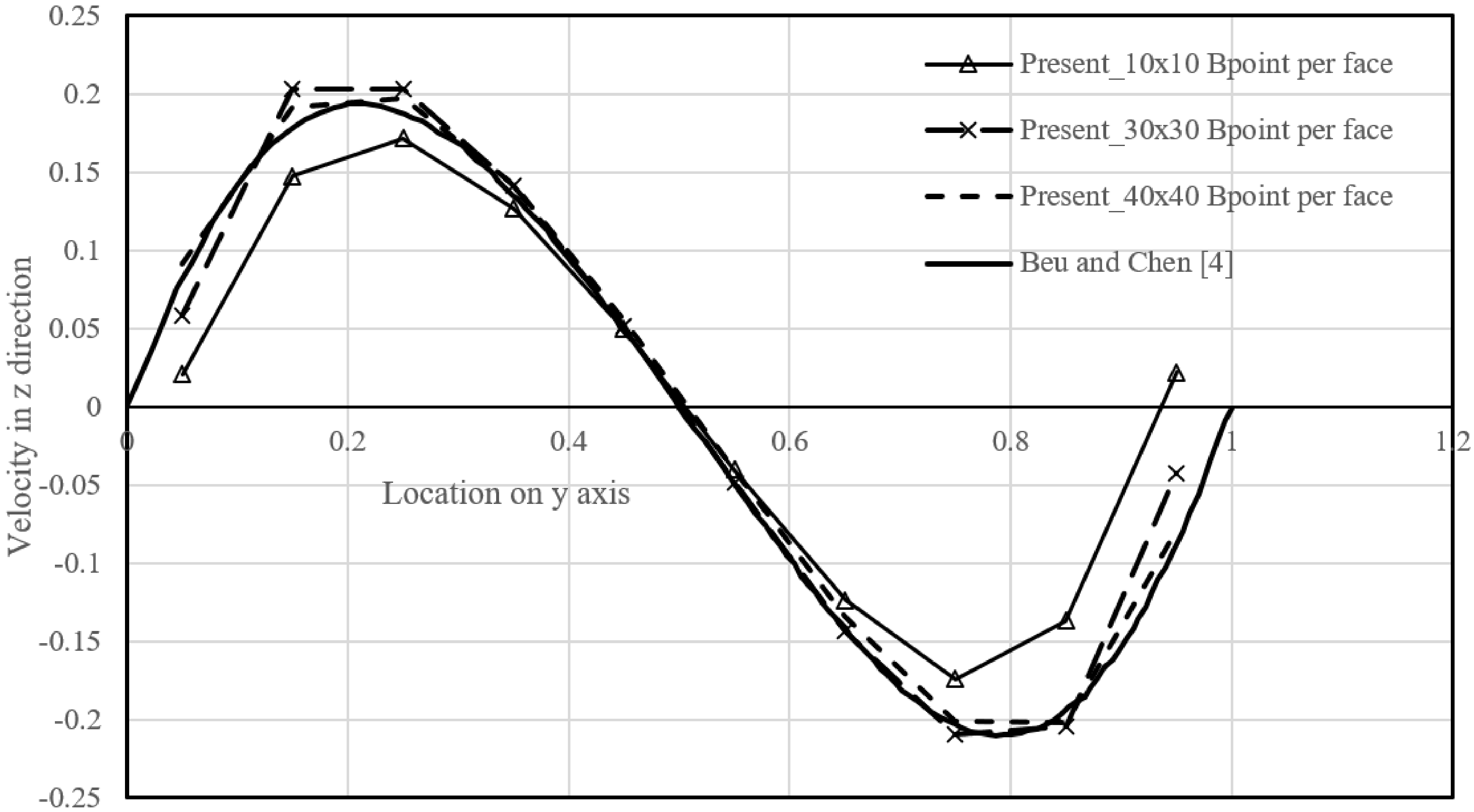
Example

Lid driven cavity

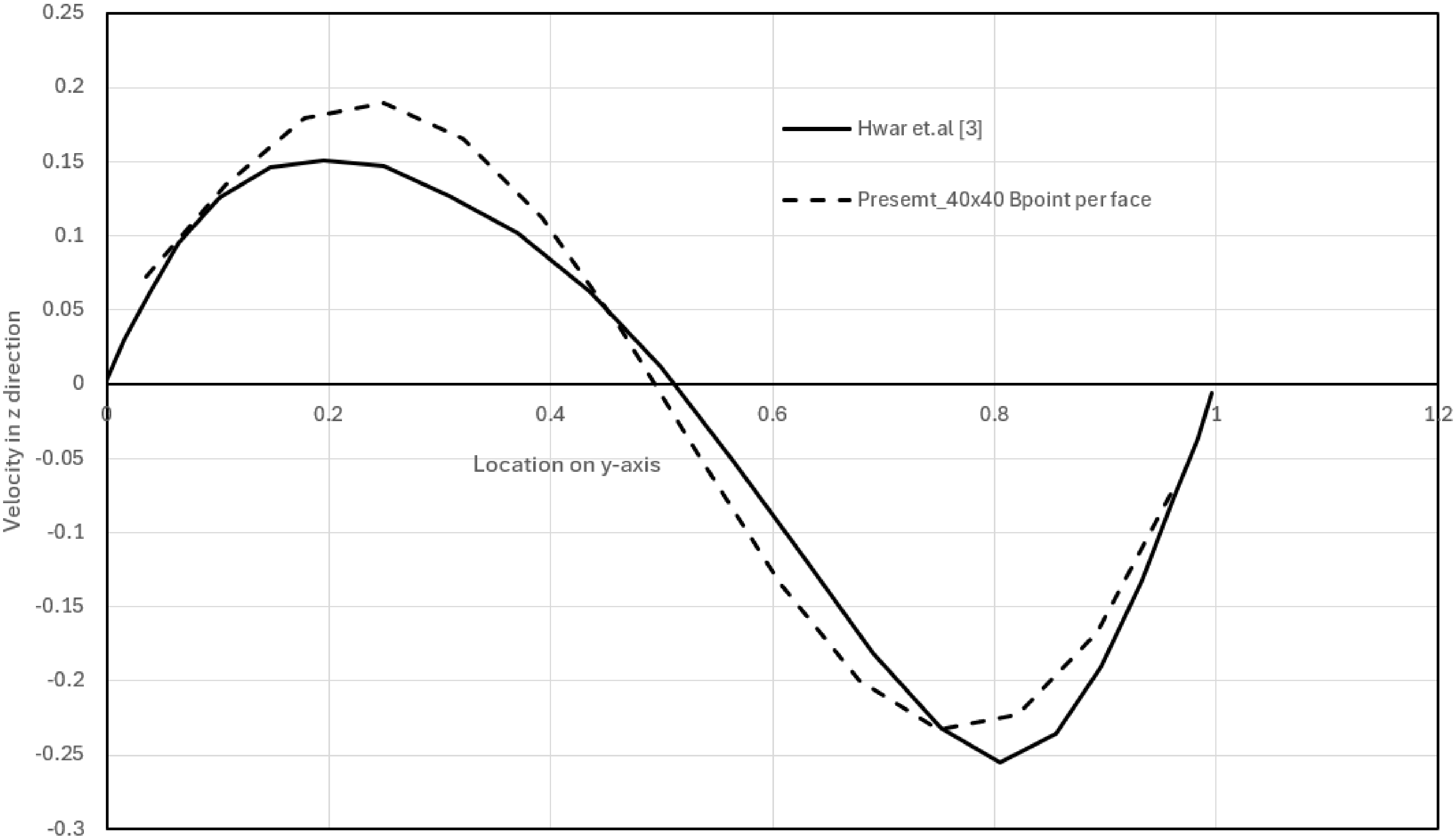


Reynolds number 10 and 100
Poisson's ratio 0.4999999
Relaxation parameter 0.05

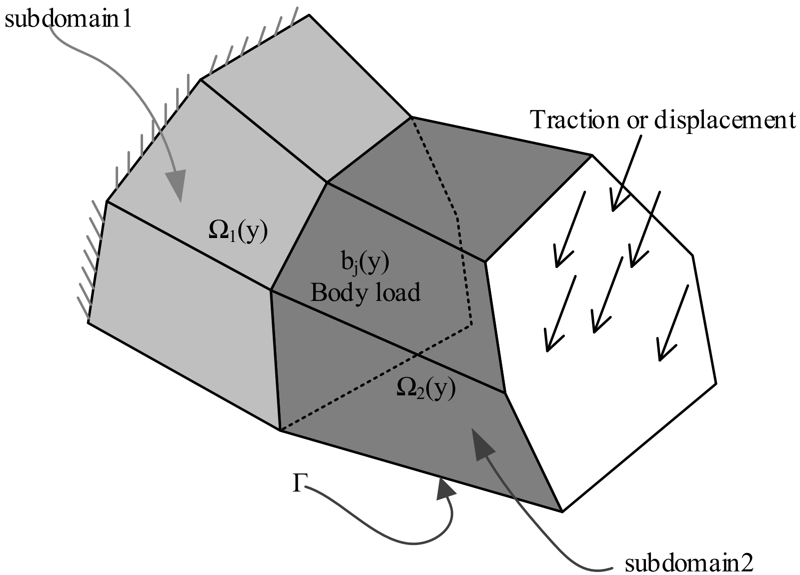
Re 10 with 10x10x10 MC
Offset 100%

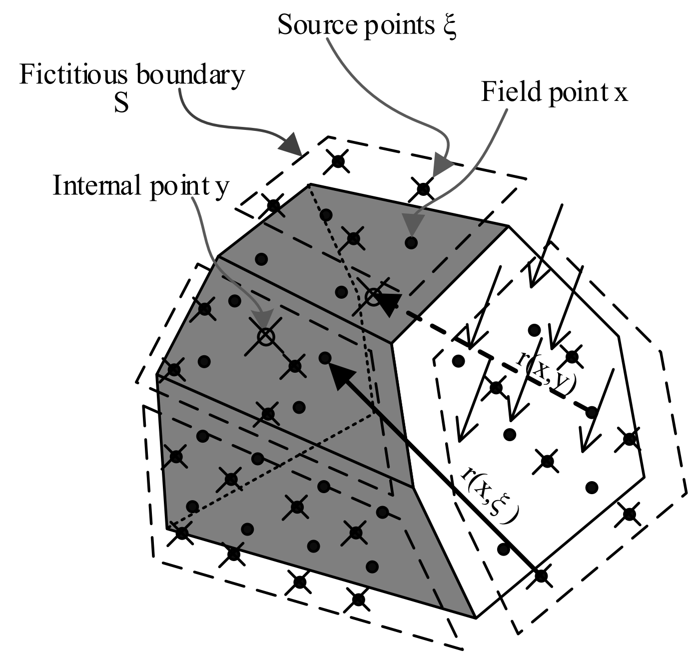


Re 100 with 14x14x14 MC
Offset 300%



Meshless MFS for multi domain





For each sub domain

Equation for displacement:

$$u_j(\mathrm{x}) = \sum_{k=1}^{k=\mathrm{N}} U_{ij}^*(\xi_k, \mathrm{x})\phi_i(\xi_k) + \int_{\Omega(\mathrm{y})} U_{ij}^*(\mathrm{y}, \mathrm{x}) b_i(y) d\Omega(\mathrm{y})$$

Equation for traction:

$$t_j(\mathrm{x}) = \sum_{k=1}^{k=\mathrm{N}} T_{ij}^*(\xi_k, \mathrm{x})\phi_i(\xi_k) + \int_{\Omega(\mathrm{y})} T_{ij}^*(\mathrm{y}, \mathrm{x}) b_i(y) d\Omega(\mathrm{y})$$

For each subdomain:

$$[F]_{3\mathrm{N_r}\times 3\mathrm{N_r}} = [K]_{3\mathrm{N_r}\times 3\mathrm{N_r}}\{u\}_{3\mathrm{N_r}\times 1} - [K]_{3\mathrm{N_r}\times 3\mathrm{N_r}}\{u^p\}_{3\mathrm{N_r}\times 1} + [F^p]_{3\mathrm{N_r}\times 3\mathrm{N_r}}$$

$$[K]_{3\mathrm{N_r}\times 3\mathrm{N_r}} = [L]_{3\mathrm{N_r}\times 3\mathrm{N_r}}[T^*]_{3\mathrm{N_r}\times 3\mathrm{N_r}}[U^*]^{-1}{}_{3\mathrm{N_r}\times 3\mathrm{N_r}}$$

Assembly:

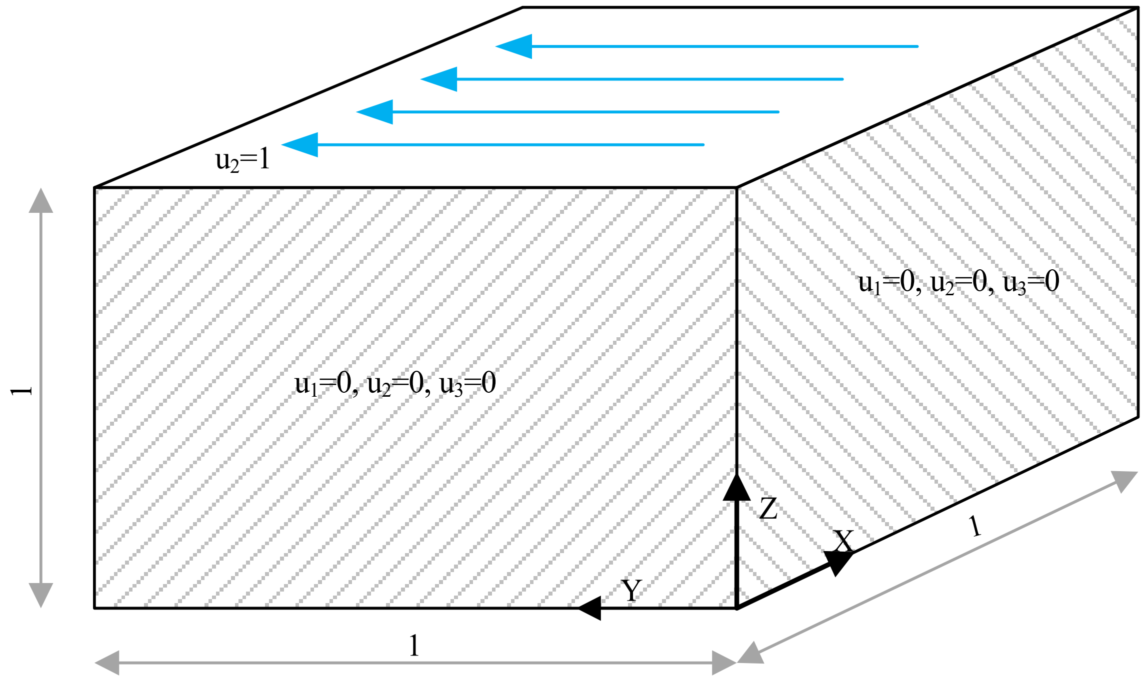
$$\begin{Bmatrix}\{F^N\}\\ \{F^U\}\end{Bmatrix}_{3\mathrm{N}\times 1} = \begin{bmatrix}[K_{11}] & [K_{12}]\\ [K_{21}] & [K_{22}]\end{bmatrix}_{3\mathrm{N}\times 3\mathrm{N}}\begin{Bmatrix}\{u^U\}\\ \{u^N\}\end{Bmatrix}_{3\mathrm{N}\times 1} - [K]_{3\mathrm{N}\times 3\mathrm{N}}\{u^p\}_{3\mathrm{N}\times 1} + \{F^p\}_{3\mathrm{N}\times 1}$$

Rearrange:

$$\begin{bmatrix}-[K_{11}] & [0]\\ -[K_{21}] & [I]\end{bmatrix}_{3\mathrm{N}\times 3\mathrm{N}}\begin{Bmatrix}\{u^U\}\\ \{F^U\}\end{Bmatrix}_{3\mathrm{N}\times 1} = \begin{bmatrix}-[I] & [K_{12}]\\ [0] & [K_{22}]\end{bmatrix}_{3\mathrm{N}\times 3\mathrm{N}}\begin{Bmatrix}\{F^N\}\\ \{u^N\}\end{Bmatrix}_{3\mathrm{N}\times 1} + \{b^p\}_{3\mathrm{N}\times 1}$$

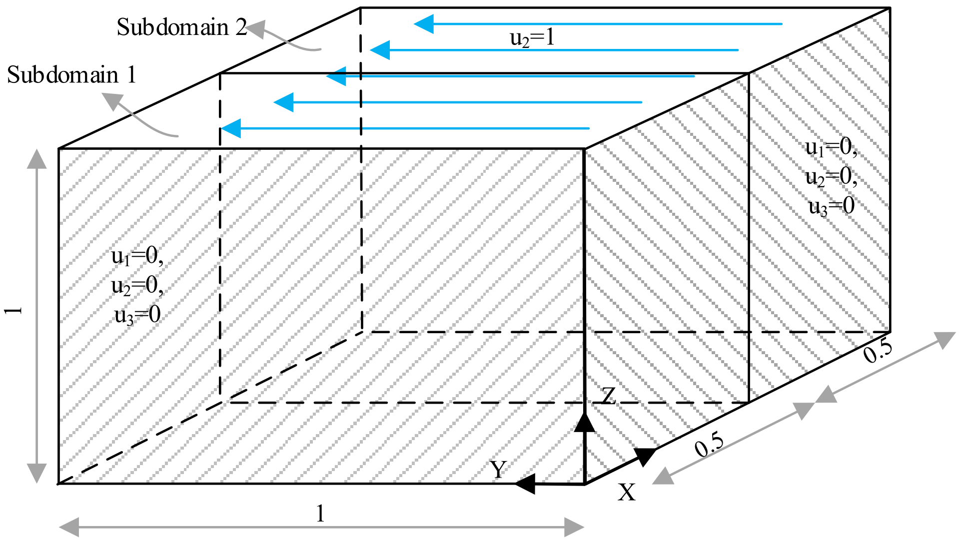
Example

Lid driven cavity



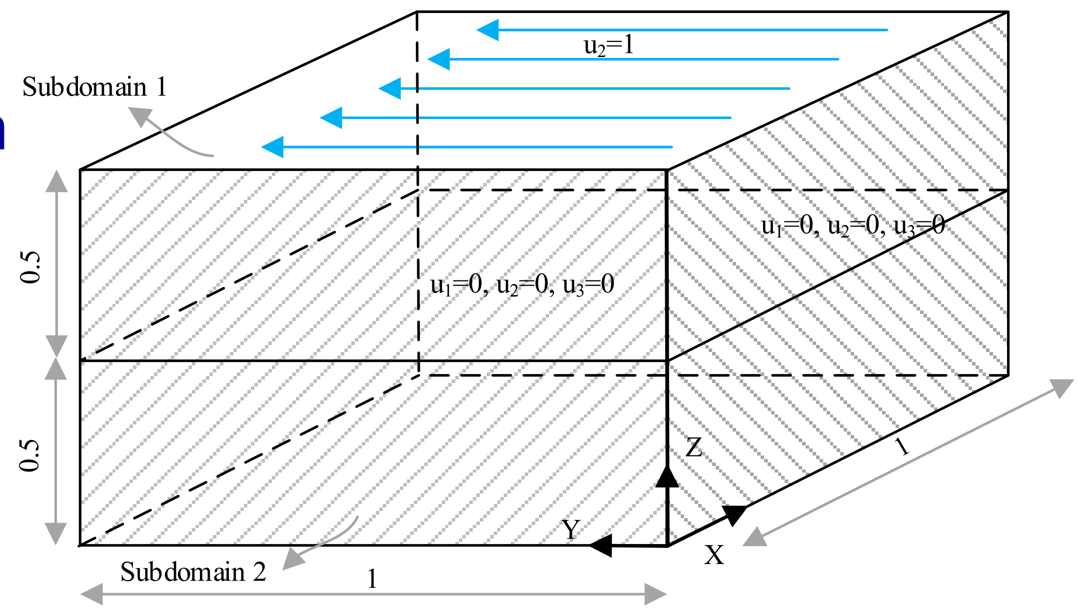
Reynolds number 10 and 100
Poisson’s ratio 0.4999999
Relaxation parameter 0.05

Example

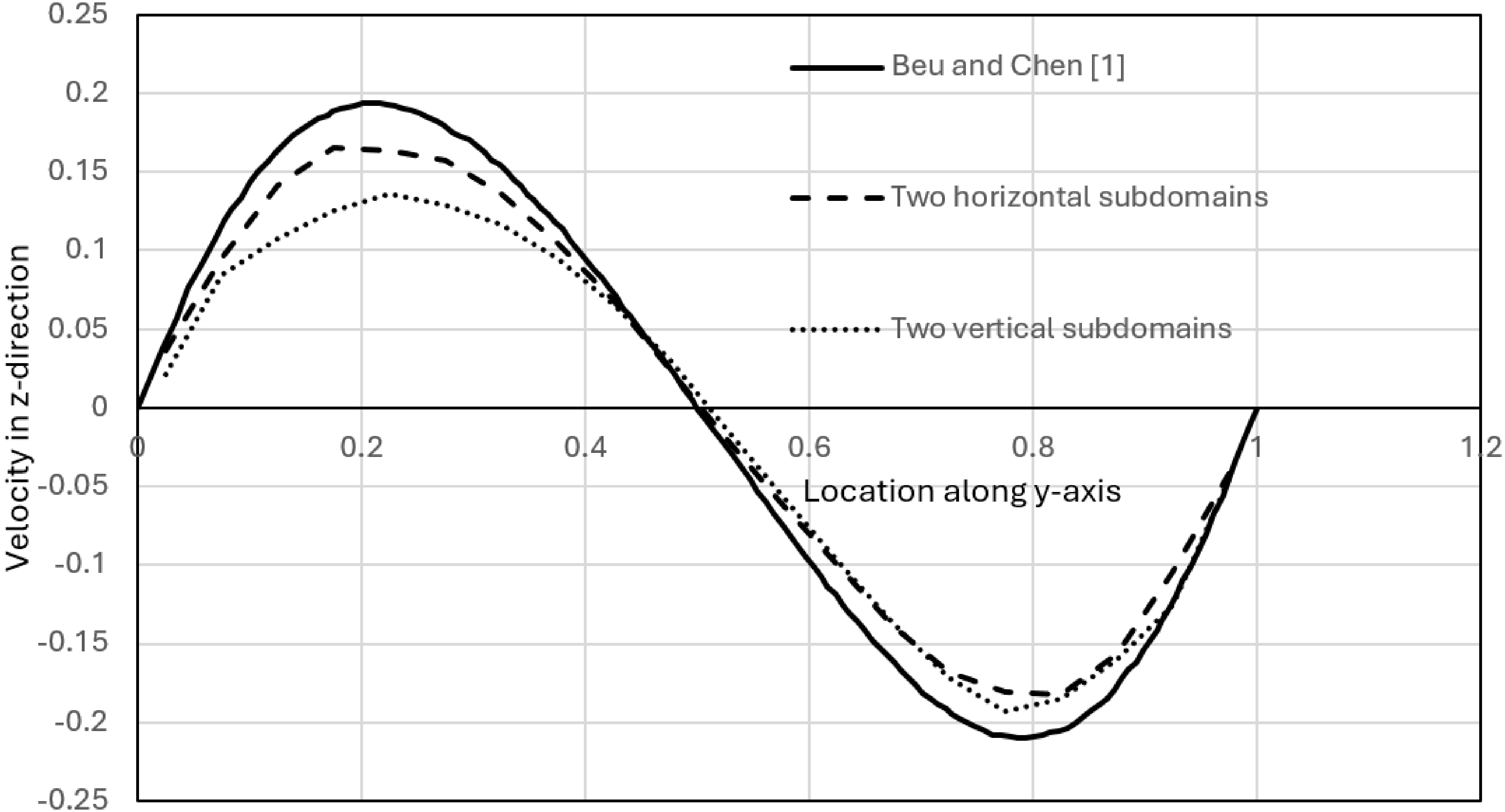


Two vertical subdomain

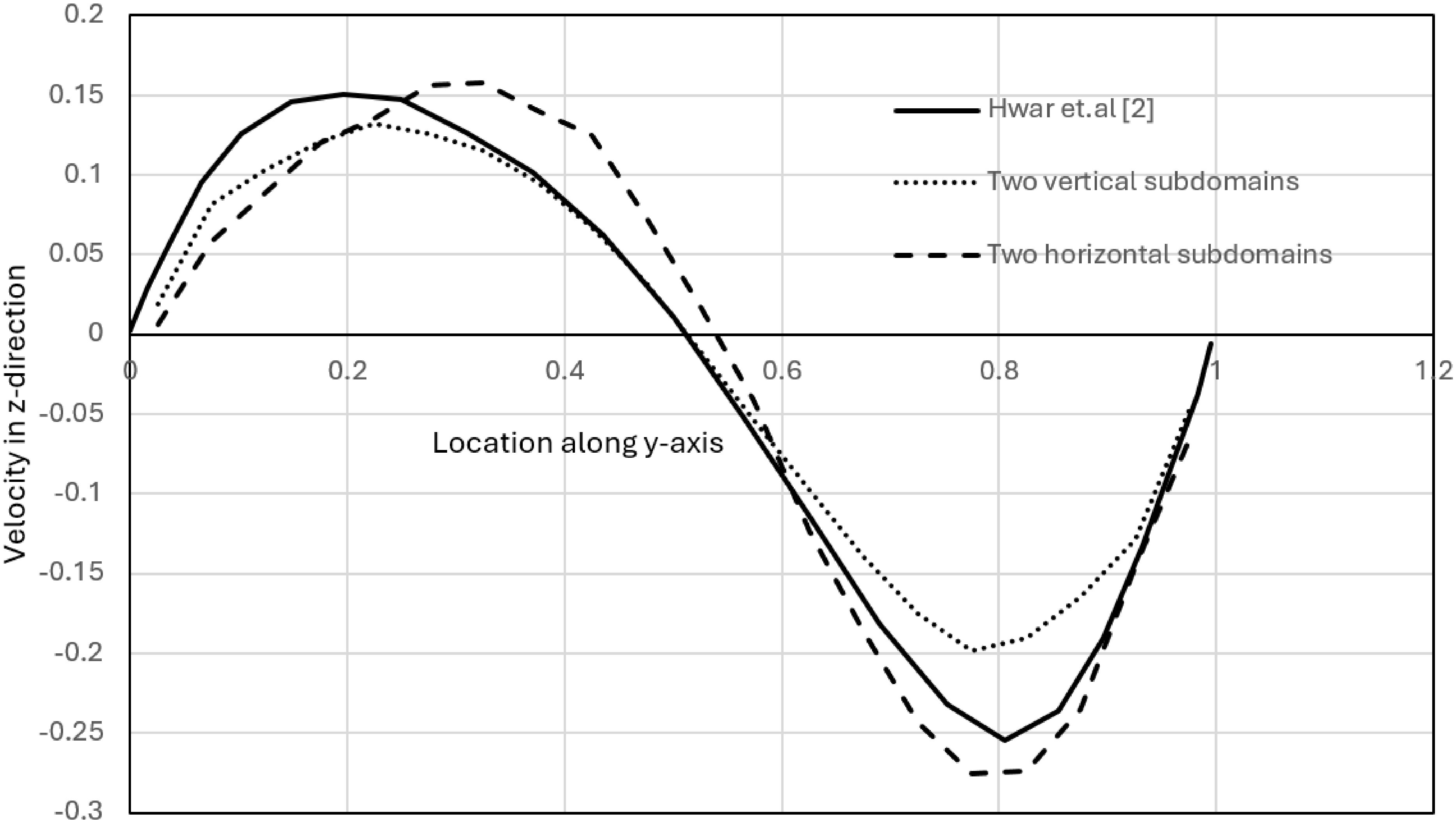
Two horizontal subdomain



Re 10 20x20 BP/face 24x24x24 MC
Offset 0.125 inner and 0.15 outer for the Vert.
Offset 0.21 for the Horz.



Re 100 20x20 BP/face 24x24x24 MC
Offset 0.125 inner and 0.15 outer for the Vert.
Offset 0.21 for the Horz.



Unsteady state CFD for 3D problems

Time differencing technique

The N-S equation at time $t+\Delta t$:

$$(\lambda+\mu)u_{j,ij}^{t+\Delta t}(\mathrm{x})+\mu u_{i,jj}^{t+\Delta t}(\mathrm{x})=-b_i^{t+\Delta t}(\mathrm{x})+\frac{\partial u_i^{t+\Delta t}}{\partial t}(\mathrm{x})$$

Using Houbolt finite difference scheme:

$$\frac{\partial}{\partial t}u_i^{t+\Delta t}=\frac{11u_i^{t+\Delta t}-18u_i^{t}+9u_i^{t-\Delta t}-2u_i^{t-2\Delta t}}{6\,\Delta t}$$

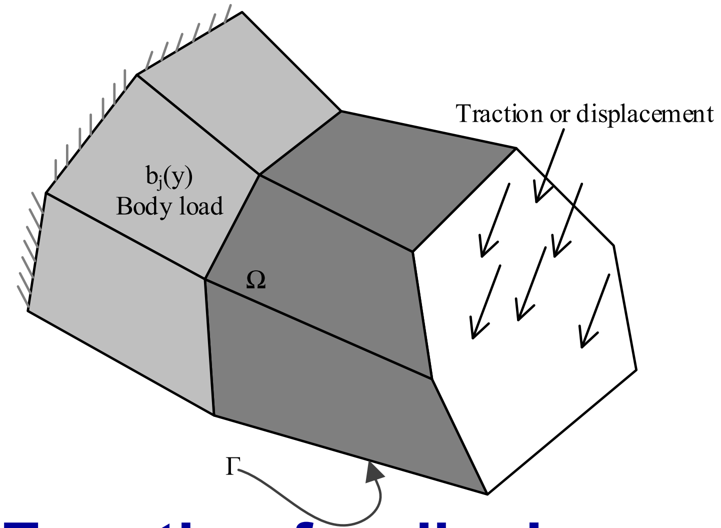
The governing differential equation can be written as follows:

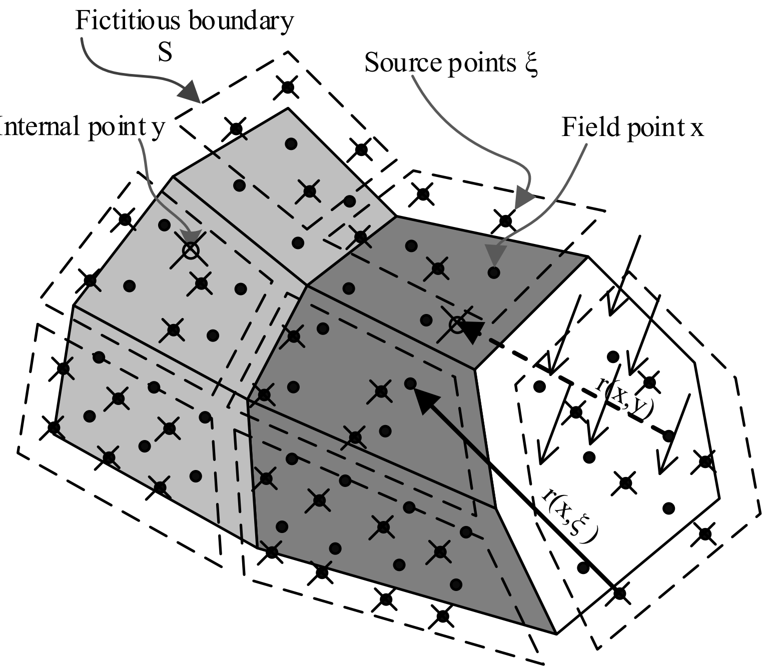
$$L_{ij}u_j^{t+\Delta t}(\mathrm{x})=-B_i^{t+\Delta t}(\mathrm{x})-b_i^{t+\Delta t}(\mathrm{x})$$

$$L_{ij}=\mu\left(\nabla^2\,\delta_{ij}+\frac{\partial_i\partial_j}{1-2\nu}-\frac{11}{6\,\Delta t\mu}\delta_{ij}\right)$$

$$B_i^{t+\Delta t}(\mathrm{x})=\frac{18u_i^{t}-9u_i^{t-\Delta t}+2u_i^{t-2\Delta t}}{6\,\Delta t}$$

Meshless MFS





Equation for displacement:

$$u_j^{t+\Delta t}(\mathrm{x}) = \sum_{k=1}^{k=\mathrm{N}} U_{ij}^{*}(\xi_k,\mathrm{x})\phi_i^{t+\Delta t}(\xi_k) + \int_{\Omega(\mathrm{y})} U_{ij}^{*}(\mathrm{y},\mathrm{x})\{B_i^{t+\Delta t}(\mathrm{y}) + b_i^{t+\Delta t}(\mathrm{y})\}d\Omega(\mathrm{y})$$

Equation for traction:

$$t_j^{t+\Delta t}(\mathrm{x}) = \sum_{k=1}^{k=\mathrm{N}} T_{ij}^{*}(\xi_k,\mathrm{x})\phi_i^{t+\Delta t}(\xi_k) + \int_{\Omega(\mathrm{y})} T_{ij}^{*}(\mathrm{y},\mathrm{x})\{B_i^{t+\Delta t}(\mathrm{y}) + b_i^{t+\Delta t}(\mathrm{y})\}d\Omega(\mathrm{y})$$

Fundamental solution for 3D

Governing equation

$$L_{ij}U^*_{jk}(\xi, \mathrm{x}) = -\delta(\xi, \mathrm{x})\delta_{ik}$$

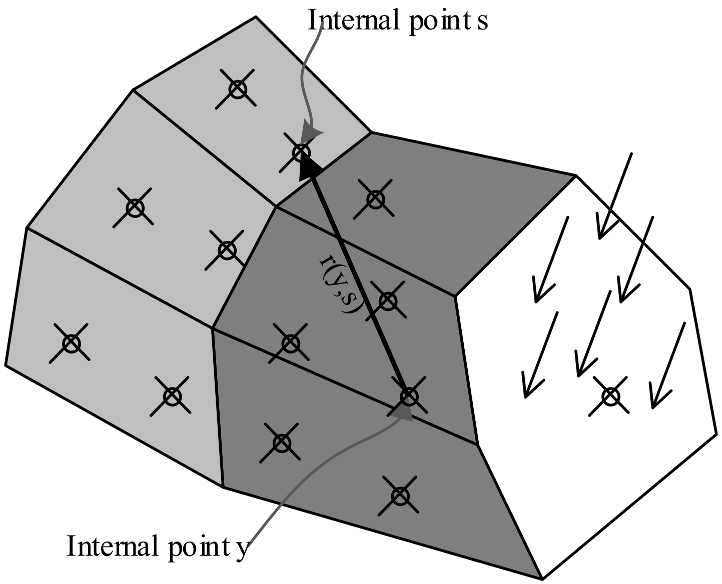
Displacement fundamental solution:

$$\begin{aligned}
&U^*_{ij}(\xi, \mathrm{x}) \\
&= \frac{-1}{8\pi\mu(1-v)\left({C_1}^2 - {C_2}^2\right)r^3}\Big[\delta_{ij}\left[e^{-C_1 r} - e^{-C_2 r}\right. \\
&\left.+ r\{C_1 e^{-C_1 r} - C_2 e^{-C_2 r} + r{C_1}^2 e^{-C_1 r}\}\right] \\
&+ r_{,i}\, r_{,j}\left[-3e^{-C_1 r} + 3e^{-C_2 r}\right. \\
&\left.+ r\{-3C_1 e^{-C_1 r} + 3C_2 e^{-C_2 r} - r{C_1}^2 e^{-C_1 r} + r{C_2}^2 e^{-C_2 r}\}\right]\Big]
\end{aligned}$$

$$C_1^2 = \frac{11}{6\,\Delta t\mu} \qquad C_2^2 = \frac{1-2\nu}{2(1-\nu)}\, C_1^2$$

Treatment of domain integral

Using Monte-Carlo integration technique



$$\int_{\Omega(\mathrm{y})} U_{ij}^{*}(\mathrm{y},\mathrm{x})\left\{B_i^{t+\Delta t}(y) + b_i^{t+\Delta t}(y)\right\} d\Omega(\mathrm{y})$$

$$= \frac{\Omega}{\mathrm{N}_m} \sum_{m=1}^{m=\mathrm{N}_m} U_{ij}^{*}(\mathrm{y}_m,\mathrm{x})\left\{B_i^{t+\Delta t}(\mathrm{y}_m) + b_i^{t+\Delta t}(\mathrm{y}_m)\right\}$$

Solution procedure:

- **Initially compute the matrices .**

At time step $t+\Delta t$

- **Compute the time dependent term from previous velocities at times (**t**,** $t-\Delta t$ **and** $t-2\Delta t$**):**

$$B_i^{t+\Delta t}(\mathrm{x}) = \frac{18u_i^t - 9u_i^{t-\Delta t} + 2u_i^{t-2\Delta t}}{6\,\Delta t}$$

- **Start the loop on the nonlinear term convective terms. At step *l*, compute the total domain term .**
- **Solve the system of equations and get** $\phi^{t+\Delta t}$ **the get the boundary velocity and traction.**

$$\begin{Bmatrix} \left\{\bar{u}^{t+\Delta t}(\mathrm{x})\right\}_{3\mathrm{N}_1\times 1} \\ \left\{\bar{t}^{t+\Delta t}(\mathrm{x})\right\}_{3\mathrm{N}_2\times 1} \end{Bmatrix} = \begin{bmatrix} [U^*(\xi,\mathrm{x})]_{3\mathrm{N}_1\times 3\mathrm{N}} \\ [T^*(\xi,\mathrm{x})]_{3\mathrm{N}_2\times 3\mathrm{N}} \end{bmatrix} \left\{\phi^{t+\Delta t}(\xi)\right\}_{3\mathrm{N}\times 1} + \left\{b_p^{t+\Delta t}(\mathrm{x})\right\}_{3\mathrm{N}\times 1}$$

Solution procedure:

- **Compute the internal velocity and its derivative.**
- **Compute the convective term:**

$$b_i{}^{t+\Delta t}(\mathrm{y}) = \mathrm{K}\left(b_i{}^{t+\Delta t}(\mathrm{y})\right)^{(l)} + (\mathrm{K}-1)\left(b_i{}^{t+\Delta t}(\mathrm{y})\right)^{(l-1)}$$

- **Compute the difference in results of internal velocities.**

$$diff = \frac{\left(u_j^{(l)} - u_j^{(l-1)}\right)100}{u_j^{(l-1)}}$$

The End

Thanks for your kind attention