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An Integral Equation Based Stiffness Formulation for Shear-Deformable Plate Bending Dynamics

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Introduction

- The dynamic analysis of plates plays an important role in several engineering fields, including civil, mechanical, and aerospace engineering.
- Deriving analytical solutions to the governing partial differential equations can be complex, numerical methods are essential for practical applications.
- Numerical methods:
  - Finite element method (FEM).
  - Boundary element method (BEM).
    - Direct method.
    - Indirect method.
    - The variational formulation.

Introduction

Dynamic analysis techniques using BEM:

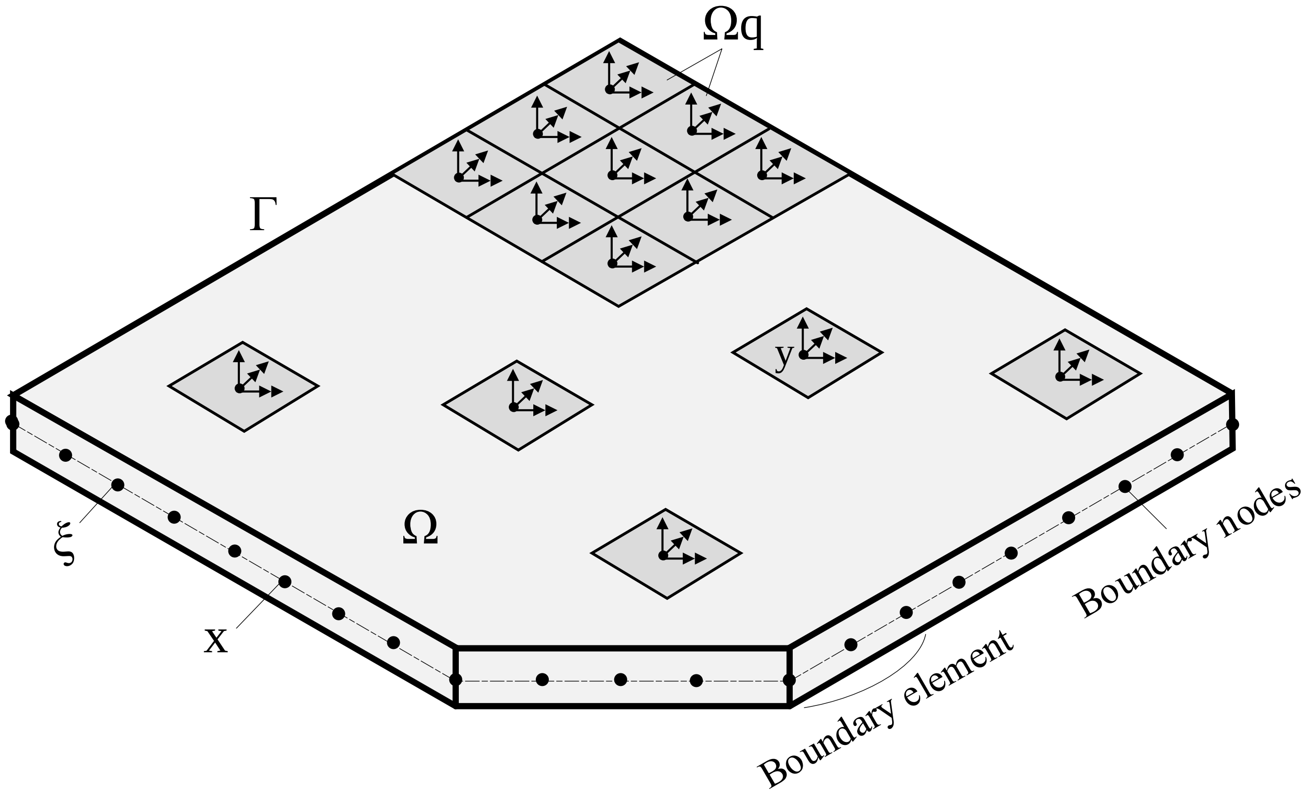
- Static fundamental solution.
- Dual reciprocity method (DRM).
- Time-differencing technique.
- Direct differencing technique:
  - Harmonic analysis.
  - Time dependent fundamental solution.
  - Laplace domain.

Main Objectives & Organization

- This research presents an innovative approach to the dynamic analysis of plate bending problems.
- The BIEM is used to form stiffness matrix and mass matrix for any arbitrary shear-deformable plate in bending using innovative technique.
- Hence those matrices are used to solve the dynamic equation of motion for the considered plate as a single super element.
- The present formulation is considered for free and forced vibration analyses.
- The developed technique involves boundary discretization plus few internal discretization to define mass and loading/measurement points.
- Several numerical examples are solved to demonstrate the accuracy and the efficiency of the present formulation.

The Proposed Dynamic Formulation

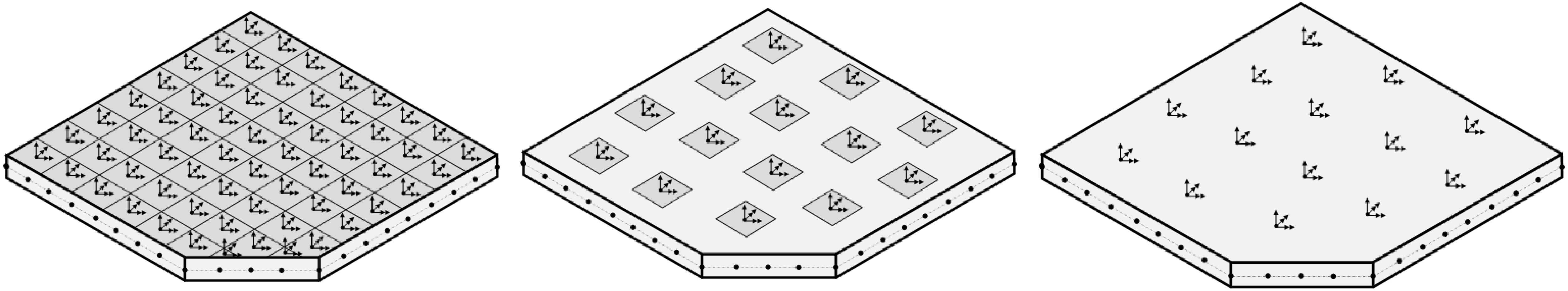
- To perform the proposed dynamic analysis of an arbitrary plate in bending, (considered herein as a super plate bending element), it is essential to extract the proposed super element stiffness matrix and mass matrix.



The proposed formulation plate geometry.

The Proposed Dynamic Formulation

- The proposed three ways of postulation DOF for an arbitrary plate bending super element:



a) Discretizing the overall plate domain into DOF cells.

b) Discretizing part of the domain points to define DOF cells.

c) Choosing small areas or points at which DOF are defined.

The Proposed Dynamic Formulation

- The direct boundary integral equation can be presented as follows:

$$C_{ij}(\xi)u_j(\xi) + \int_{\Gamma(x)} T_{ij}(\xi, x)u_j(x)\, d\Gamma(x) = \int_{\Gamma(x)} U_{ij}(\xi, x)t_j(x)\, d\Gamma(x)$$

$$+ \int_{\Gamma(x)} K_{i3}(\xi, x)\, P_3(x) d\Gamma(x) + \sum_{N_q} \int_{\Omega_q(y)} W_{ik}(\xi, y)\, F_k^q(y) d\Omega_q(y)$$

- By collocating at the center point of each DOF cells, an additional integral equation could be rewritten:

$$u_j(Y) + \int_{\Gamma(x)} T_{ij}(Y, x)u_j(x)\, d\Gamma(x) = \int_{\Gamma(x)} U_{ij}(Y, x)t_j(x)\, d\Gamma(x)$$

$$+ \int_{\Gamma(x)} K_{i3}(Y, x)\, P_3(x) d\Gamma(x) + \sum_{N_q} \int_{\Omega_q(y)} W_{ik}(Y, y)\, F_k^q(y) d\Omega_q(y)$$

Where:

$$W_{ik}(\xi, c) = U_{ik}(\xi, c) - \frac{\upsilon}{(1-\upsilon)\lambda^2} U_{i\alpha,\alpha}(\xi, c)\delta_{3k}$$

$$W_{ik}(\xi, y) = U_{ik}(\xi, y) - \frac{\upsilon}{(1-\upsilon)\lambda^2} U_{i\alpha,\alpha}(\xi, y)\delta_{3k}.$$

The Proposed Dynamic Formulation

- Integral equations can be written in a matrix form as follows:

$$\begin{array}{c|c|c|c} & 3N & 3N_q & 3N_q \\ \hline 3N & [A] & [A_1] & [0] \\ 3N_q & [A_2] & [A_3] & [I] \end{array} \times \begin{array}{c|c} & 1 \\ \hline 3N & \{u/t\}^b \\ 3N_q & \{F\}^q \\ 3N_q & \{u\}^q \end{array} = \begin{array}{c|c} & 1 \\ \hline 3N & \{b\}^b \\ 3N_q & \{b\}^q \end{array} + \begin{array}{c|c} & 1 \\ \hline 3N & \{RHS\}^b \\ 3N_q & \{RHS\}^q \end{array}$$

The Proposed Dynamic Formulation

- **The proposed derivation of [K]**

  In order to compute [K], matrix form is used with different stiffness cases. Each stiffness case involves applying a unit deformation in one of the three degrees of freedom (DOFs) of the stiffness cell, with domain loading is set to zero.

$$\overset{\quad 3N \qquad\quad 3N_q}{\begin{array}{c} 3N \\ 3N_q \end{array}\left[\begin{array}{c|c} [A] & [A_1] \\ \hline [A_2] & [A_3] \end{array}\right]} \times \overset{1}{\begin{array}{c} 3N \\ 3N_q \end{array}\left[\begin{array}{c} \{u/t\}^b \\ \hline \{F\}^q \end{array}\right]} = \overset{1}{\begin{array}{c} 3N \\ 3N_q \end{array}\left[\begin{array}{c} \{b\}^b \\ \hline \{b\}^q - \{u\}^q \end{array}\right]}$$

Where:

$[F]^{q}_{3N_q\times 1}$ represents one load case corresponding to prescribed virtual displacement cases.

$[u]^{q}_{3N_q\times 1}$ contains the prescribed virtual displacement case.

The Proposed Dynamic Formulation

Hence, the stiffness matrix could be computed directly as follows:

$$\begin{array}{c|c|c|} & 3N & 3N_q \\ \hline 3N & [A] & [A_1] \\ \hline 3N_q & [A_2] & [A_3] \\ \hline \end{array} \times \begin{array}{c|c|} & 3N_q \\ \hline 3N & [u/t]^b \\ \hline 3N_q & [F]^q \\ \hline \end{array} = \begin{array}{c|c|} & 3N_q \\ \hline 3N & [b]^b \\ \hline 3N_q & [b]^q - [I] \\ \hline \end{array}$$

Where:

$[F]^q_{3N_q \times 3N_q}$ is the required stiffness matrix $[K]^q_{3N_q \times 3N_q}$.

The Proposed Dynamic Formulation

- **The proposed derivation of [M]**

  Mass matrix represents the loads in the stiffness cells when the deformations are Zero. Consider domain loading as own weight of the plate.

$$\begin{array}{c|c|c|}
 & 3N & 3N_q \\
\hline
3N & [A] & [A_1] \\
\hline
3N_q & [A_2] & [A_3] \\
\hline
\end{array}
\times
\begin{array}{c|c|}
 & 1 \\
\hline
3N & \{u/t\}^b \\
\hline
3N_q & \{F\}^q \\
\hline
\end{array}
=
\begin{array}{c|c|}
 & 1 \\
\hline
3N & \{b\}^b \\
\hline
3N_q & \{b\}^q \\
\hline
\end{array}
+
\begin{array}{c|c|}
 & 1 \\
\hline
3N & \{RHS\}^b \\
\hline
3N_q & \{RHS\}^q \\
\hline
\end{array}$$

Where:

$\{F\}^q_{3N_q \times 1}$ represents the load vector due to own weight which represents the main diagonal of the required mass matrix $[M]^q_{3N_q \times 3N_q}$.

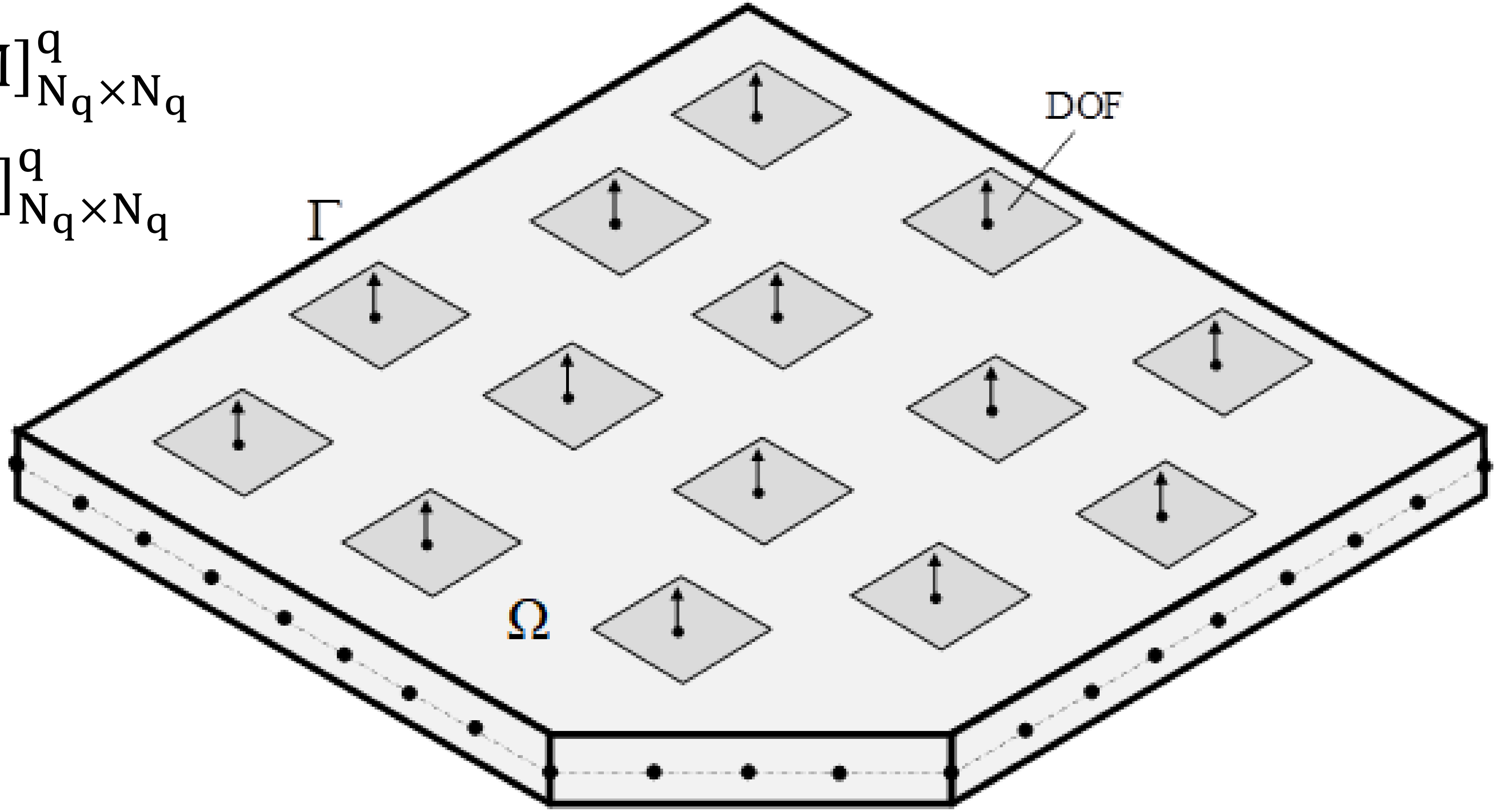
$$\{F\}^q_{3N_q \times 1} = \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_{3N_q} \end{Bmatrix} \longrightarrow [M]^q_{3N_q \times 3N_q} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & m_{3N_q} \end{bmatrix}$$

The Proposed Dynamic Formulation

Once $[K]^{q}_{3N_q \times 3N_q}$ and $[M]^{q}_{3N_q \times 3N_q}$ are established, the rotations are condensed to extract the matrices $[K]^{q}_{N_q \times N_q}$ and $[M]^{q}_{N_q \times N_q}$.

$$[M]^{q}_{3N_q \times 3N_q} \longrightarrow [M]^{q}_{N_q \times N_q}$$

$$[K]^{q}_{3N_q \times 3N_q} \longrightarrow [K]^{q}_{N_q \times N_q}$$



The Proposed Dynamic Formulation

- **Free vibration**

Consider the plate moving freely. Therefore, the equation of motion will be as follows:

$$[K]^{q}_{N_q \times N_q} \{u_3\}_{3N_q \times 1} - \omega^2 [M]^{q}_{N_q \times N_q} \{u_3\}_{3N_q \times 1} = 0$$

By computing the eigenvalues (ω^2), the corresponding natural frequencies can be determined:

$$f \frac{\omega}{2\pi}$$

The Proposed Dynamic Formulation

- **Forced vibration**

Consider the plate subjected to dynamic load p(t), the equation of motion will be as follows:

$$[M]^{q}_{N_q \times N_q} \{\ddot{u}^{t}_{3}(x)\}_{3N_q \times 1} - [K]^{q}_{N_q \times N_q} \{u^{t}_{3}(x)\}_{3N_q \times 1} = \{P^{t}_{3}(x)\}_{3N_q \times 1}$$

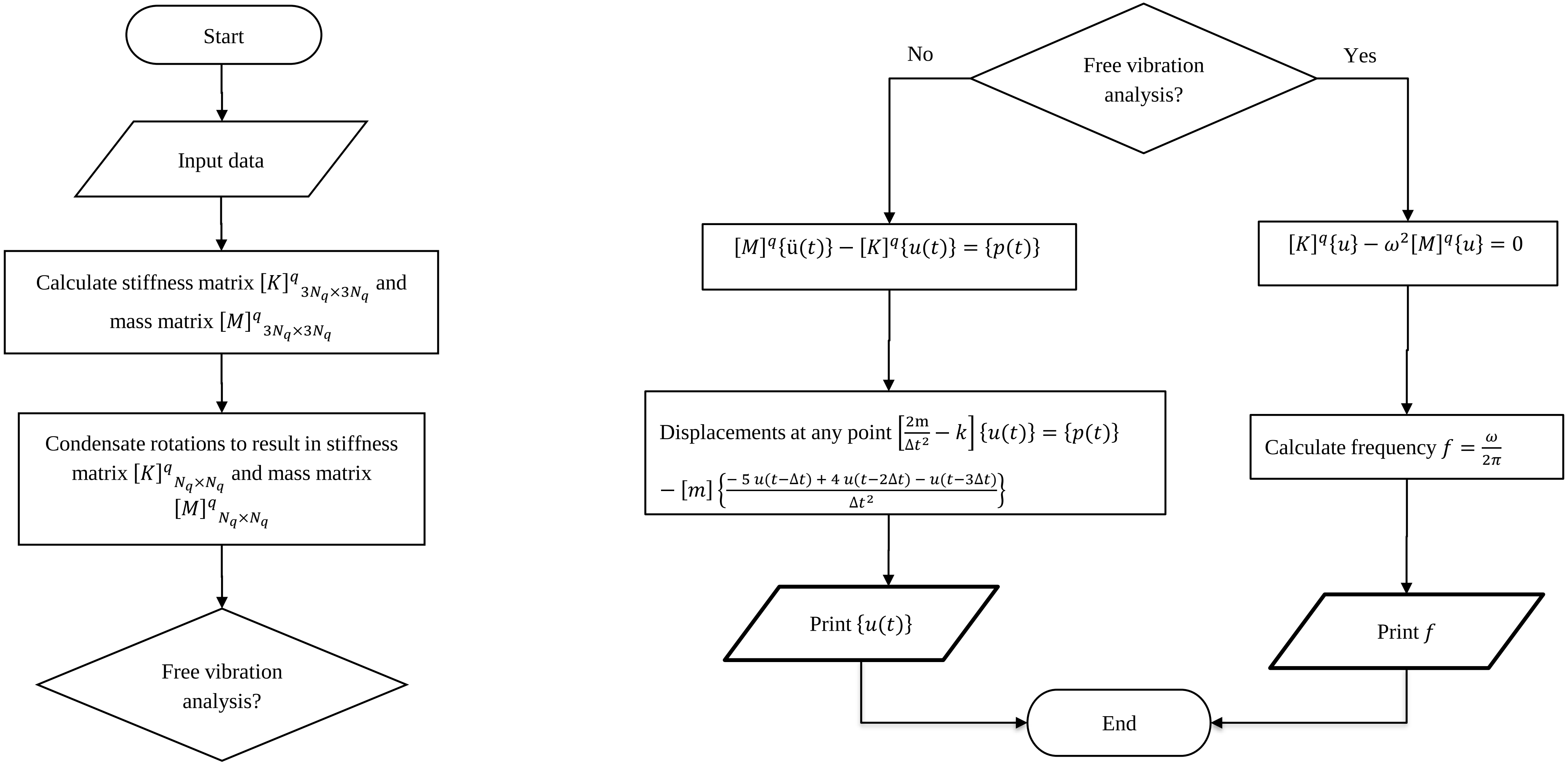
By using Houbolt scheme, the acceleration could be represented as:

$$\ddot{u}^{t}_{3}(x) = \frac{2\,u^{t}_{3}(x) - 5\,u^{t-\Delta t}_{3}(x) + 4\,u^{t-2\Delta t}_{3}(x) - u^{t-3\Delta t}_{3}(x)}{\Delta t^{2}}$$

By applying Eq.10 to the previous plate, the displacement in any point of plate could be calculated as follows:

$$\left(\frac{2[M]^{q}_{N_q \times N_q}}{\Delta t^{2}} - [K]^{q}_{N_q \times N_q}\right)\{u^{t}_{3}(x)\}_{3N_q \times 1} = \{P^{t}_{3}(x)\}_{3N_q \times 1} - [M]^{q}_{N_q \times N_q}\left(\frac{\{-5\,u^{t-\Delta t}_{3}(x) + 4\,u^{t-2\Delta t}_{3}(x) - u^{t-3\Delta t}_{3}(x)\}_{3N_q \times 1}}{\Delta t^{2}}\right)$$

Numerical Implementation



A flowchart for the dynamic analysis of a bending plate program.

Numerical Examples

- **Example.1 Free vibration of rectangular plate with simply supported along its longitudinal edges and is clamped on the transversal edges**

  Rectangular plate 4 × 10 m.

  thicknesses 1 m.

  E= $22 \times 10^5\ t/m^2$.

  ρ= 0.245

  υ=0.3.

  Results are compared with analytical solutions from:

  - *Hashemi, S. H., & Arsanjani, M. (2005). Exact characteristic equations for some of classical boundary conditions of vibrating moderately thick rectangular plates. International Journal of Solids and Structures, 42(3-4), 819-853.*
  - *Senjanović, I., Vladimir, N., & Tomić, M. (2013). An advanced theory of moderately thick plate vibrations. Journal of Sound and Vibration, 332(7), 1868-1880.*
  - *Xing, Y., & Liu, B. (2009). Characteristic equations and closed-form solutions for free vibrations of rectangular Mindlin plates. Acta Mechanica Solida Sinica, 22(2), 125-136.*

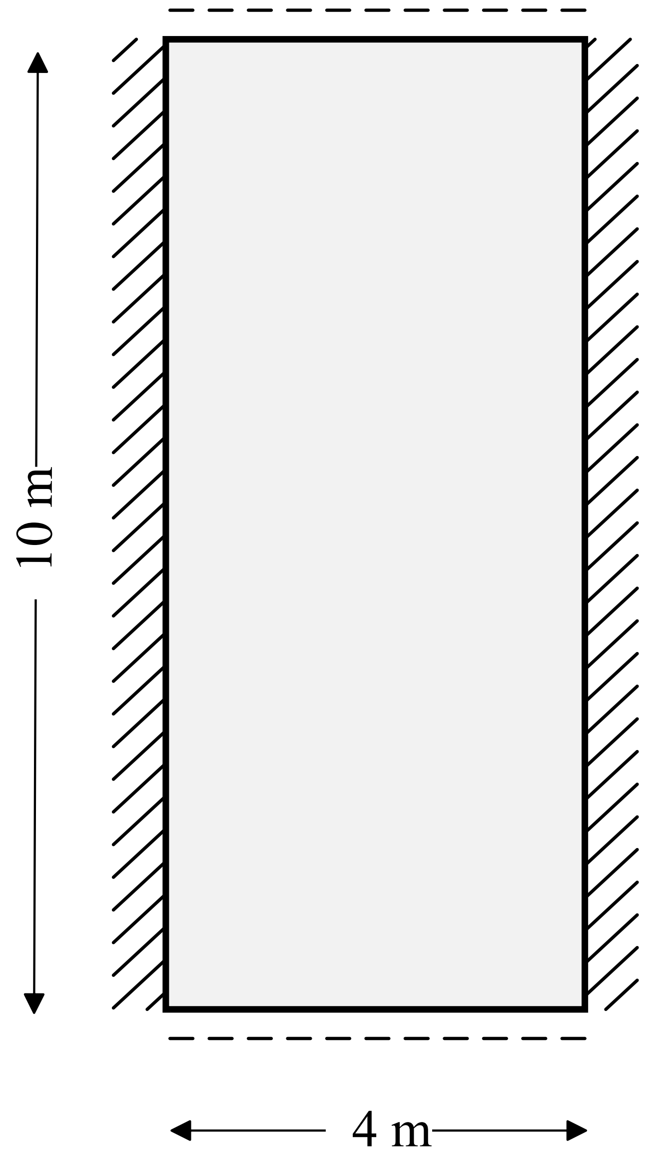


Table 1: Frequencies of the plate in example 8.1, case A.

Method	Mesh	Frequencies		
		1st mode	**2nd mode**	**3rd mode**
FEM-thick plate	10 elements – 54 DOFs	138.26	158.43	180.28
	40 elements – 165 DOFs	151.45	171.37	203.08
	160 elements – 567 DOFs	153.85	176.63	214.65
	640 elements – 2091 DOFs	154.41	178.02	217.75
	2560 elements – 8019 DOFs	154.54	178.36	218.53
FEM-3D	20 elements – 162 DOFs	189.16	191.71	202.44
	80 elements – 495 DOFs	165.99	175.53	200.96
	320 elements – 1701 DOFs	158.25	173.16	205.39
	2560 elements – 10455 DOFs	154.2	170.68	204.94
	20480 elements – 72171 DOFs	152.75	169.49	203.91
Hashemi and Arsanjani (analytical solution) [36]		151.5176	169.3645	205.3788
Senjanovic` et al. [37]		150.8543	167.3181	202.3269
xing and liu [38]		150.8543	167.3181	202.3269
The present formulation **(Full domain discretization)**	2×5 DOFs - 48 BEs	165.1568	188.9109	234.5172
	4×10 DOFs - 48 BEs	154.6794	175.5966	218.0689
	8×20 DOFs - 48 BEs	152.7133	172.6101	212.9192
	16×40 DOFs - 96 BEs	151.6171	171.3643	211.2518
	20×50 DOFs - 96 BEs	151.0668	170.8367	210.8032
The present formulation **(Partial discretization)**	40 DOFs - 24 BEs	148.1572	165.1272	197.0555
	60 DOFs - 24 BEs	149.9154	167.8261	202.5941
	90 DOFs - 24 BEs	151.0232	169.5314	206.1324
	119 DOFs - 48 BEs	151.5332	170.3326	207.8202
	160 DOFs - 48 BEs	151.8497	170.8546	208.9734

Numerical Examples

- **Example.2 Free vibration of rectangular plate with clamped on all edges**

Rectangular plate 4 × 10 m.

thicknesses 2 m.

E= $22 \times 10^5\ t/m^2$.

ρ= 0.245

υ=0.3.

Results are compared with analytical solutions from:

- *Xing, Y., & Liu, B. (2009). Characteristic equations and closed-form solutions for free vibrations of rectangular Mindlin plates. Acta Mechanica Solida Sinica, 22(2), 125-136.*

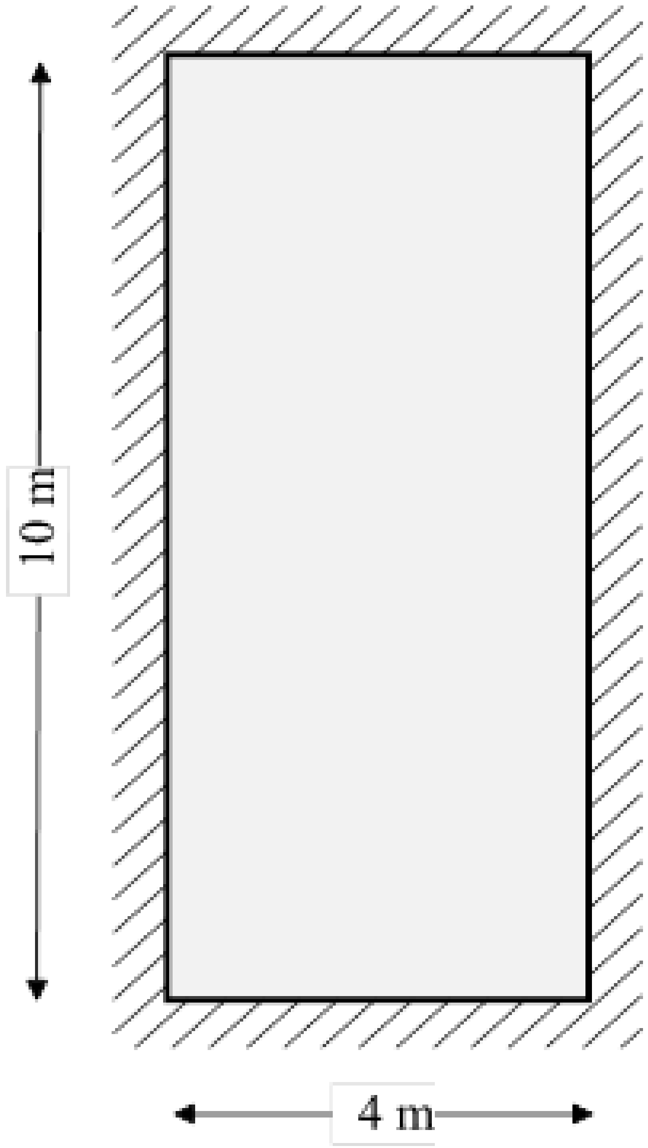


Table 2: Frequencies of the plate in example 8.1, case B.

Method	Mesh	Frequencies		
		1st mode	2nd mode	3rd mode
FEM-thick plate	10 elements – 54 DOFs	174.79	191.3	209.9
	40 elements – 165 DOFs	195.59	226.25	269.72
	160 elements – 567 DOFs	200.42	236.33	276.29
	640 elements – 2091 DOFs	201.62	238.93	277.34
	2560 elements – 8019 DOFs	201.92	239.58	277.39
FEM-3D	20 elements – 162 DOFs	204.15	208.36	231.6
	80 elements – 495 DOFs	209.53	231.85	266.63
	640 elements – 2835 DOFs	205.17	234.4	275.03
	5120 elements – 18819 DOFs	203.2	234.14	277.08
	40960 elements – 136323 DOFs	202.456	233.841	277.569
Xing and Liu [38]		190.4758	214.8479	265.8787
The present formulation **(Full domain discretization)**	2×5 DOFs - 48 BEs	212.3518	253.0886	316.1474
	4×10 DOFs - 48 BEs	204.1458	241.8413	302.9049
	8×20 DOFs - 48 BEs	201.3464	237.7801	296.2242
	16×40 DOFs - 48 BEs	198.2805	234.8915	293.1798
	20×50 DOFs - 96 BEs	194.4376	231.6125	291.0607
The present formulation **(Partial discretization)**	40 DOFs - 24 BEs	188.1857	215.7145	254.7044
	60 DOFs - 24 BEs	193.2703	223.7286	268.771
	90 DOFs - 48 BEs	196.5612	229.0343	278.5489
	119 DOFs - 48 BEs	198.0092	231.4433	283.1540
	160 DOFs - 48 BEs	199.0063	233.2192	286.6806

Numerical Examples

- **Example.3 Forced vibration analysis of simply supported plate**

Dimensions 10 × 10 × 0.5 inches.

$E = 1\times 10^{7}\ psi.$

$\upsilon = 0.3.$

$\rho = 0.259 \times 10^{-3}\ Ib.s^{2}/in^{4}$

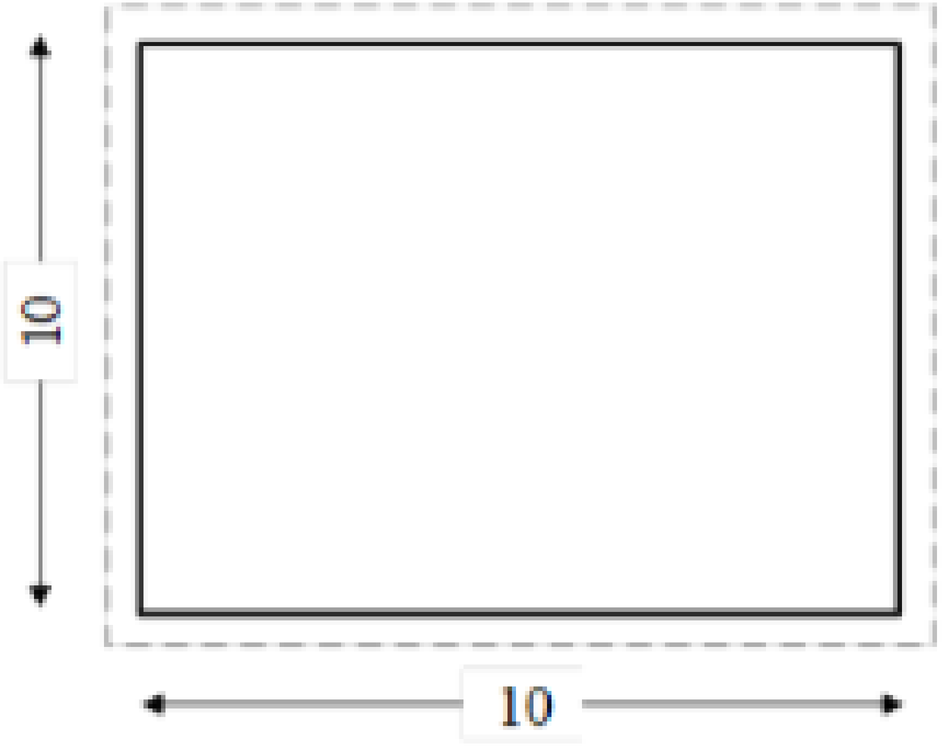
Heaviside dynamic load P(t) = 300 $psi.$

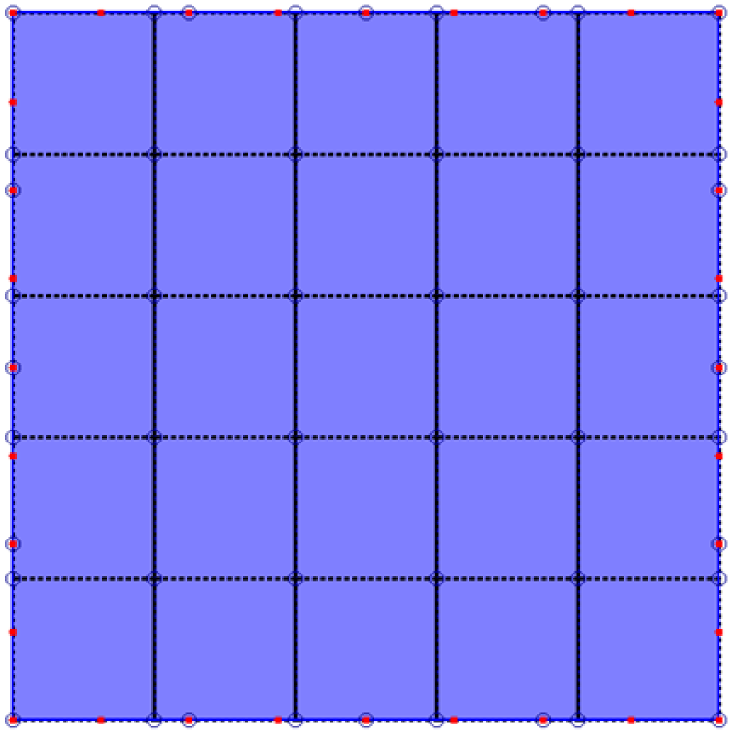
uniformly distributed over the entire area of the plate.

$\Delta t = 0.223 \times 10^{-4}\ sec.$

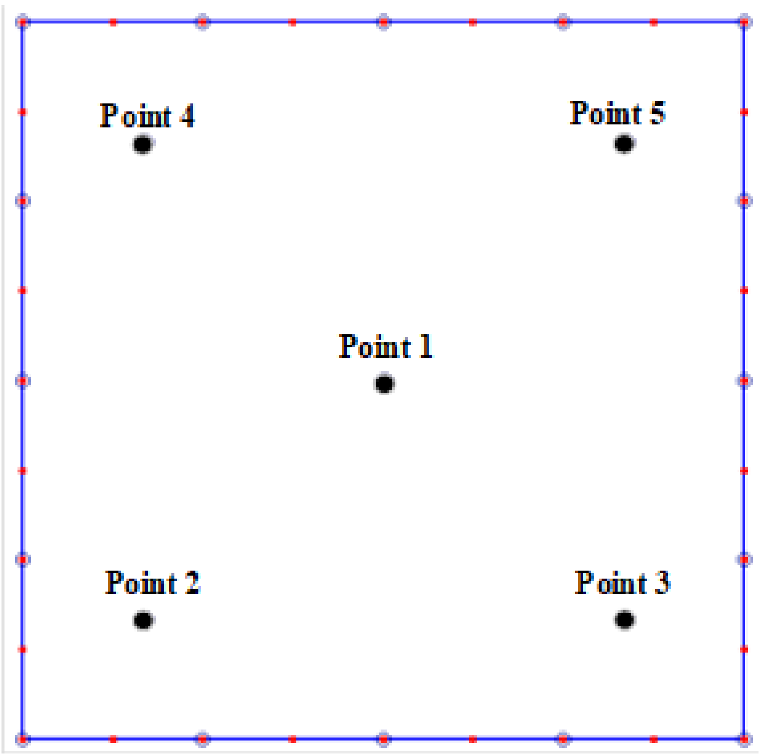
Results are compared with analytical solutions from:

- *Bauer, H. F. (1968). Nonlinear response of elastic plates to pulse excitations. Journal of Applied Mechanics, 35(1), 47–52.*

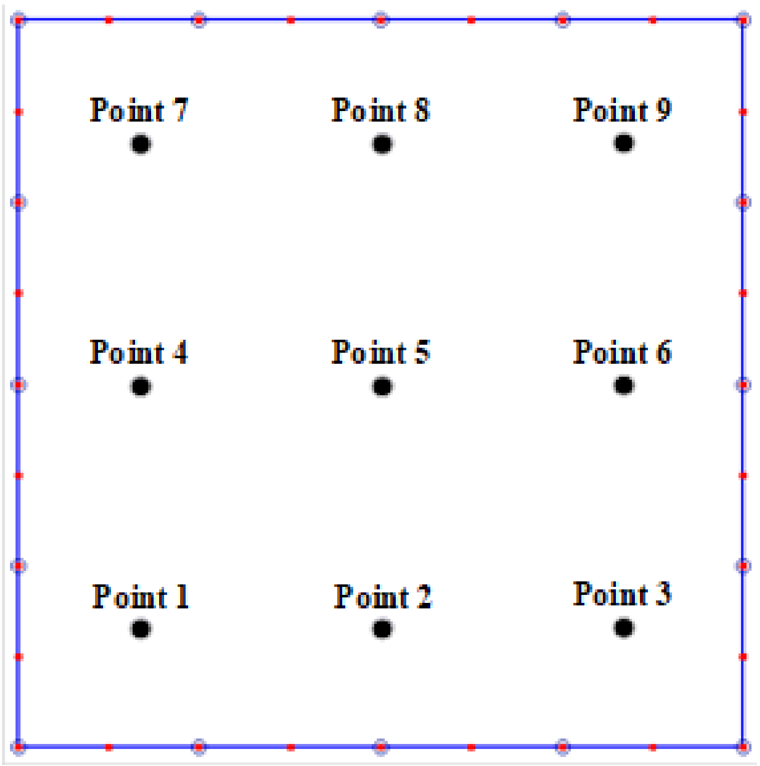




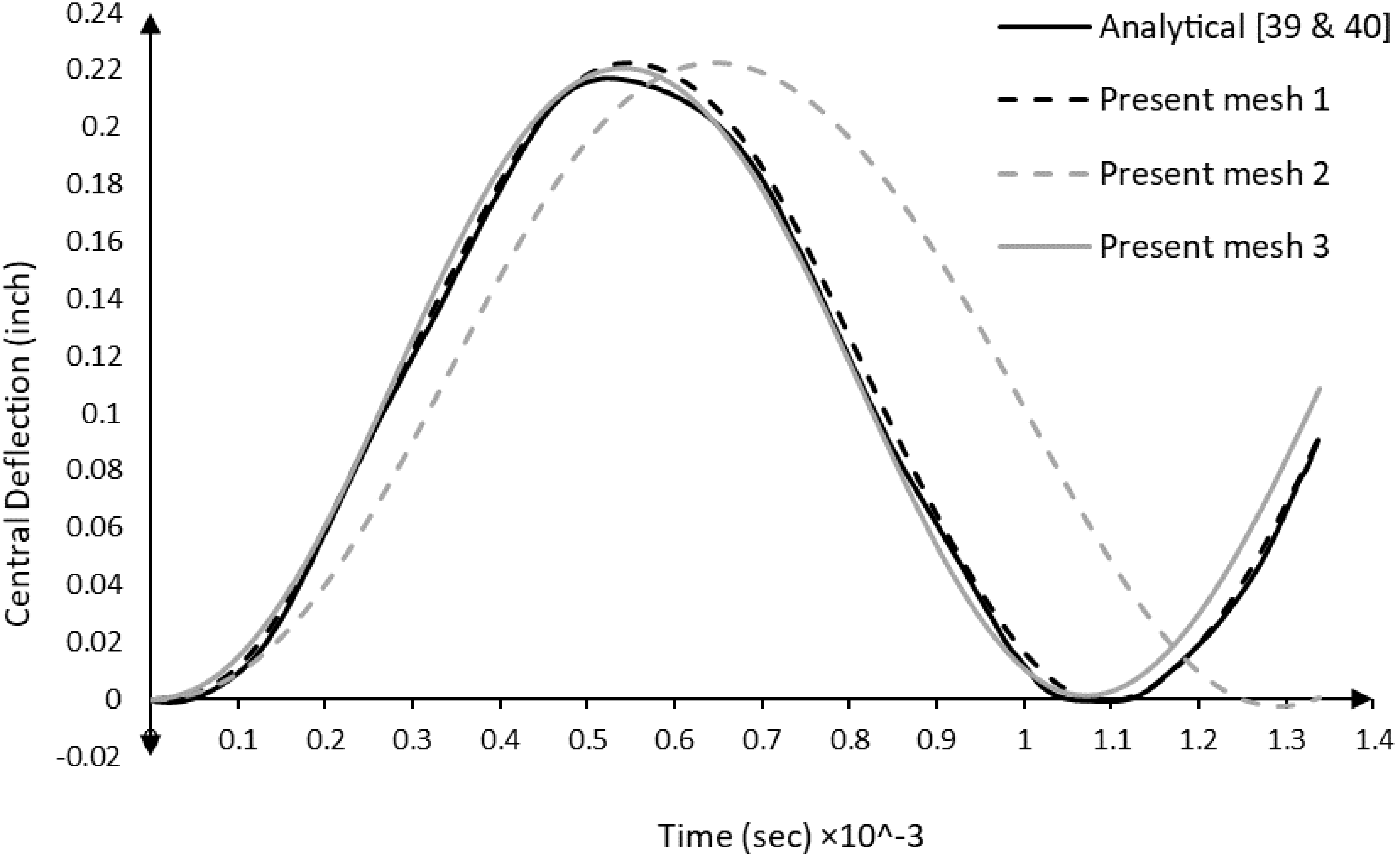
Present formulation mesh 1
(Full domain discretization)



Present formulation mesh 2
(partial discretization)



Present formulation mesh 3
(partial discretization)



Analytical [39 & 40]
Present mesh 1
Present mesh 2
Present mesh 3
Central Deflection (inch)
Time (sec) ×10^-3
0.24
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4

Numerical Examples

- **Example.4 Forced vibration analysis of simply supported plate**

Dimensions $1 \times \sqrt{2} \times 0.2$ inches.

E=1 psi.

$\upsilon = 0.3$.

ρ= 1 $Ib.s^2/in^4$.

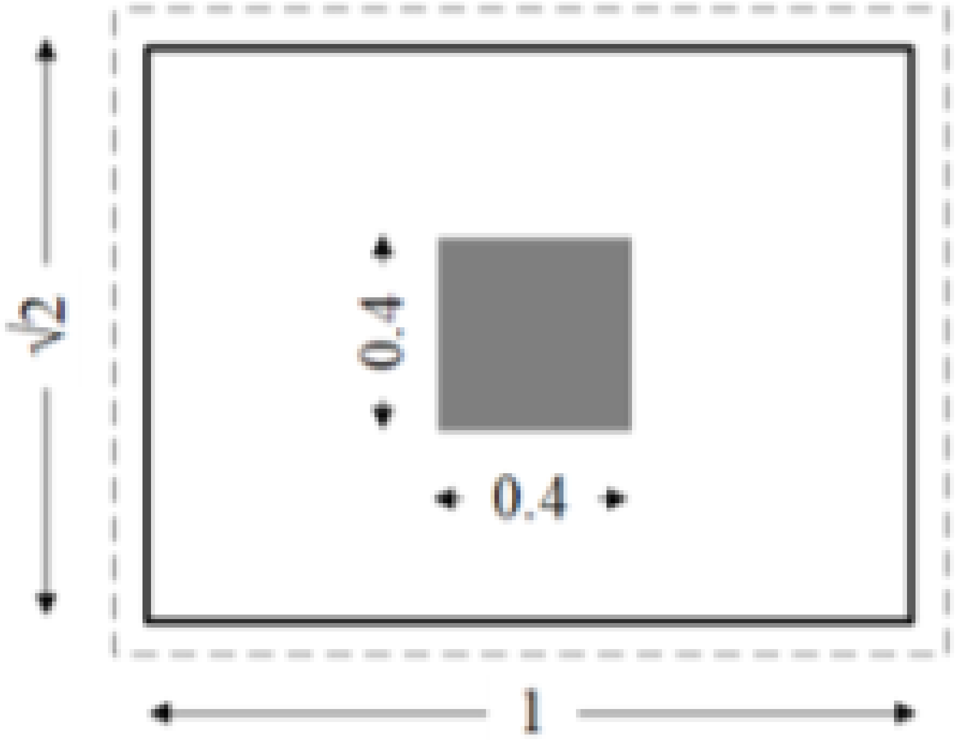
Heaviside dynamic load P(t) = $\sqrt{2}$ psi .

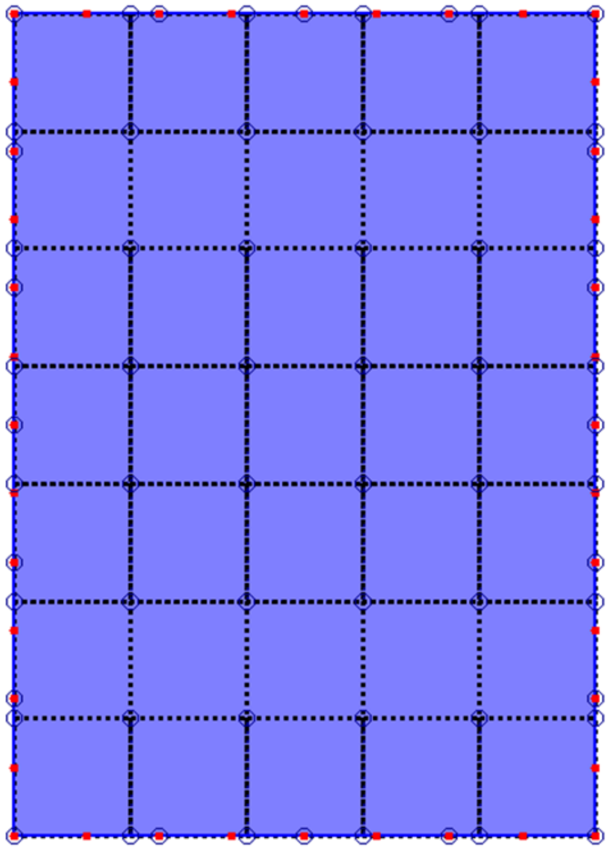
uniformly distributed over area 0.4×0.4 inches .

$\Delta t = 0.1$ sec.

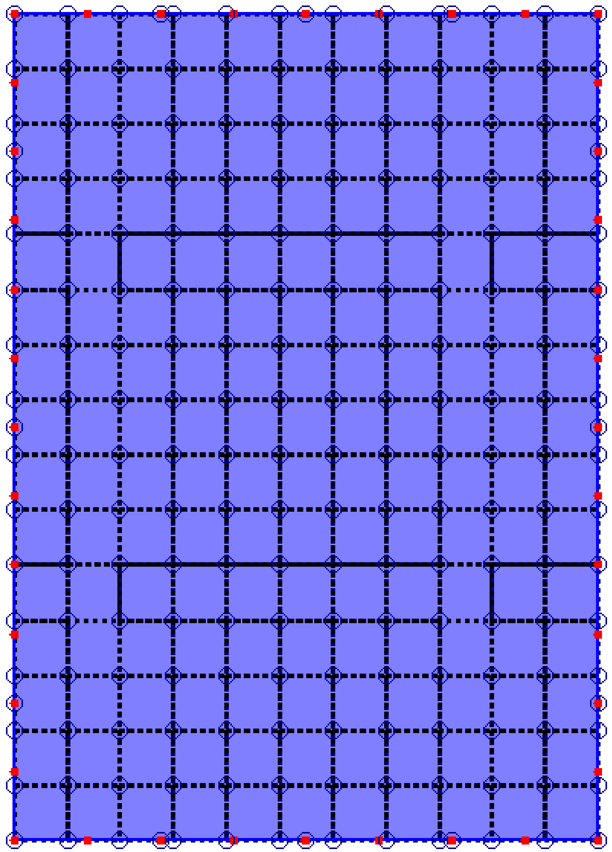
Results are compared with analytical solution from:

- *Reismann, H. E. R. B. E. R. T., & Lee, Y. (1969). Forced motion of rectangular plates. Developments in theoretical and applied mechanics, 4, 3-18.*

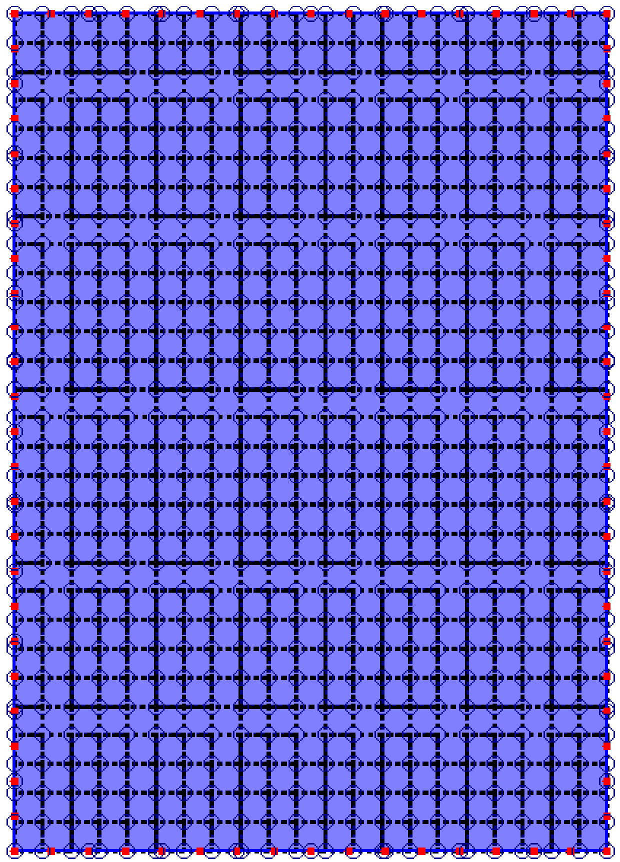




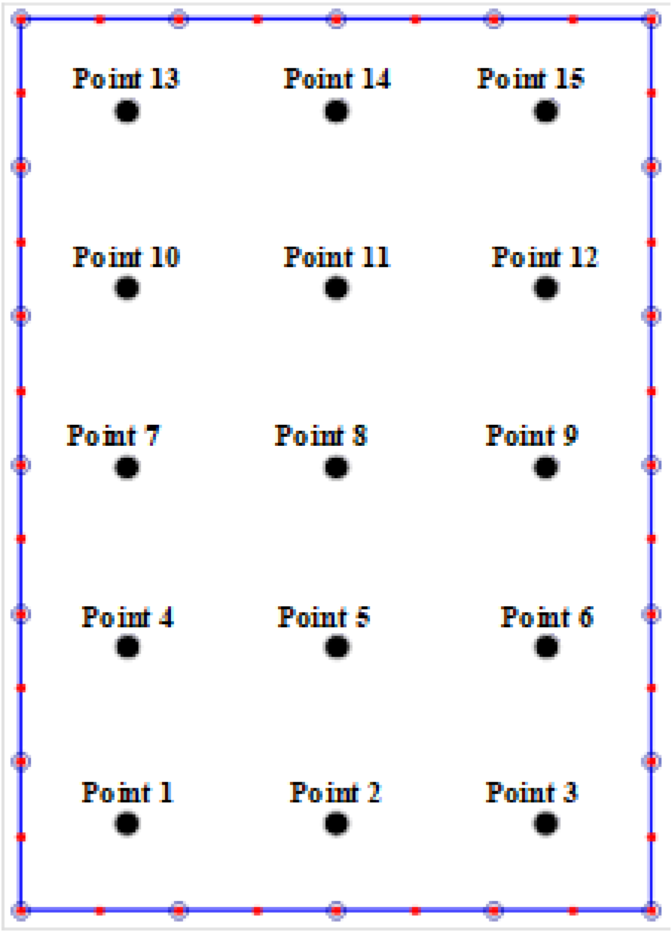
Present formulation mesh 1
(Full domain discretization)



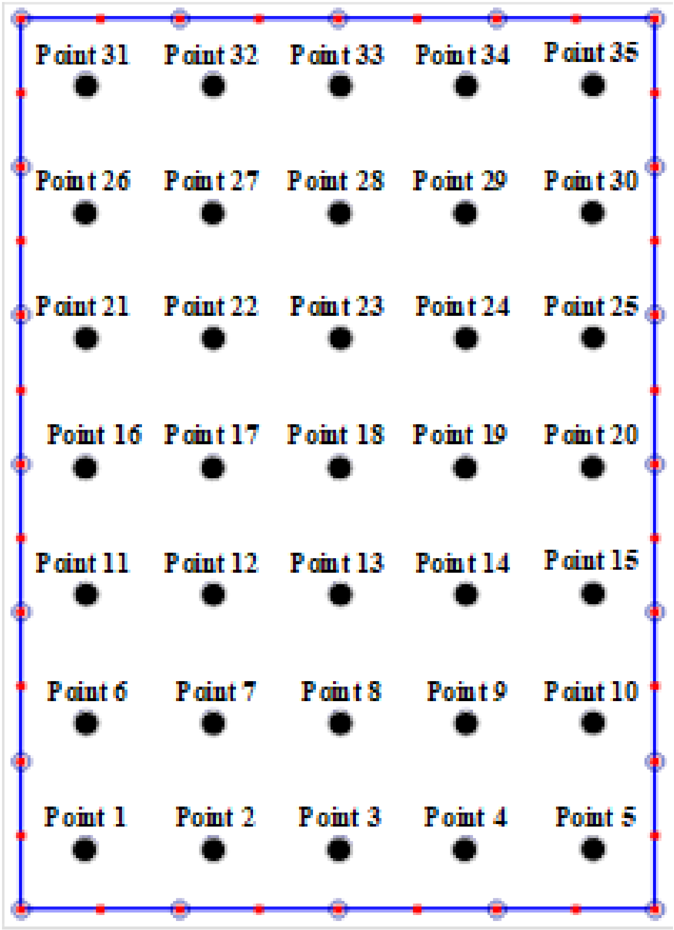
Present formulation mesh 2
(Full domain discretization)



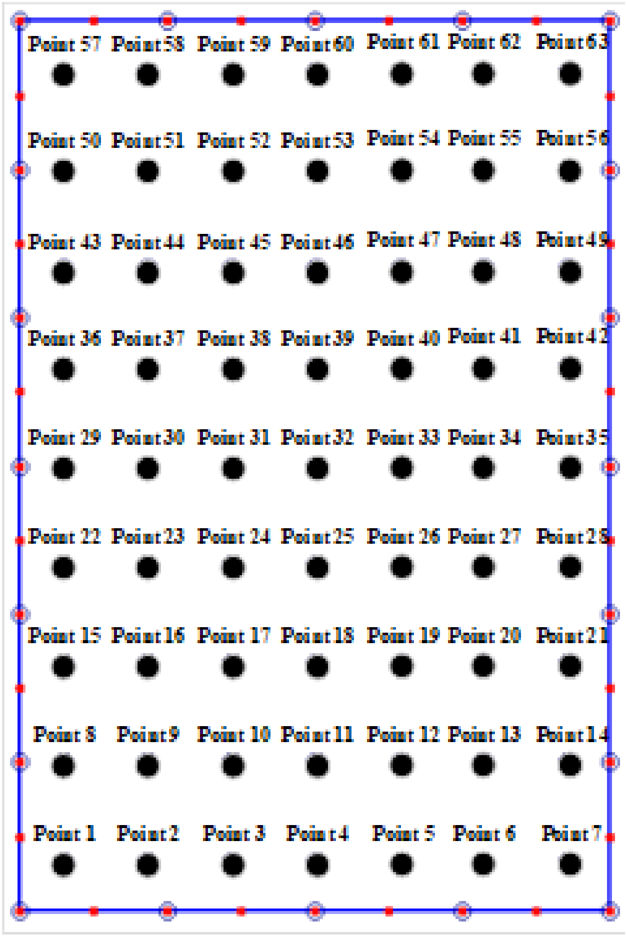
Present formulation mesh 3
(Full domain discretization)



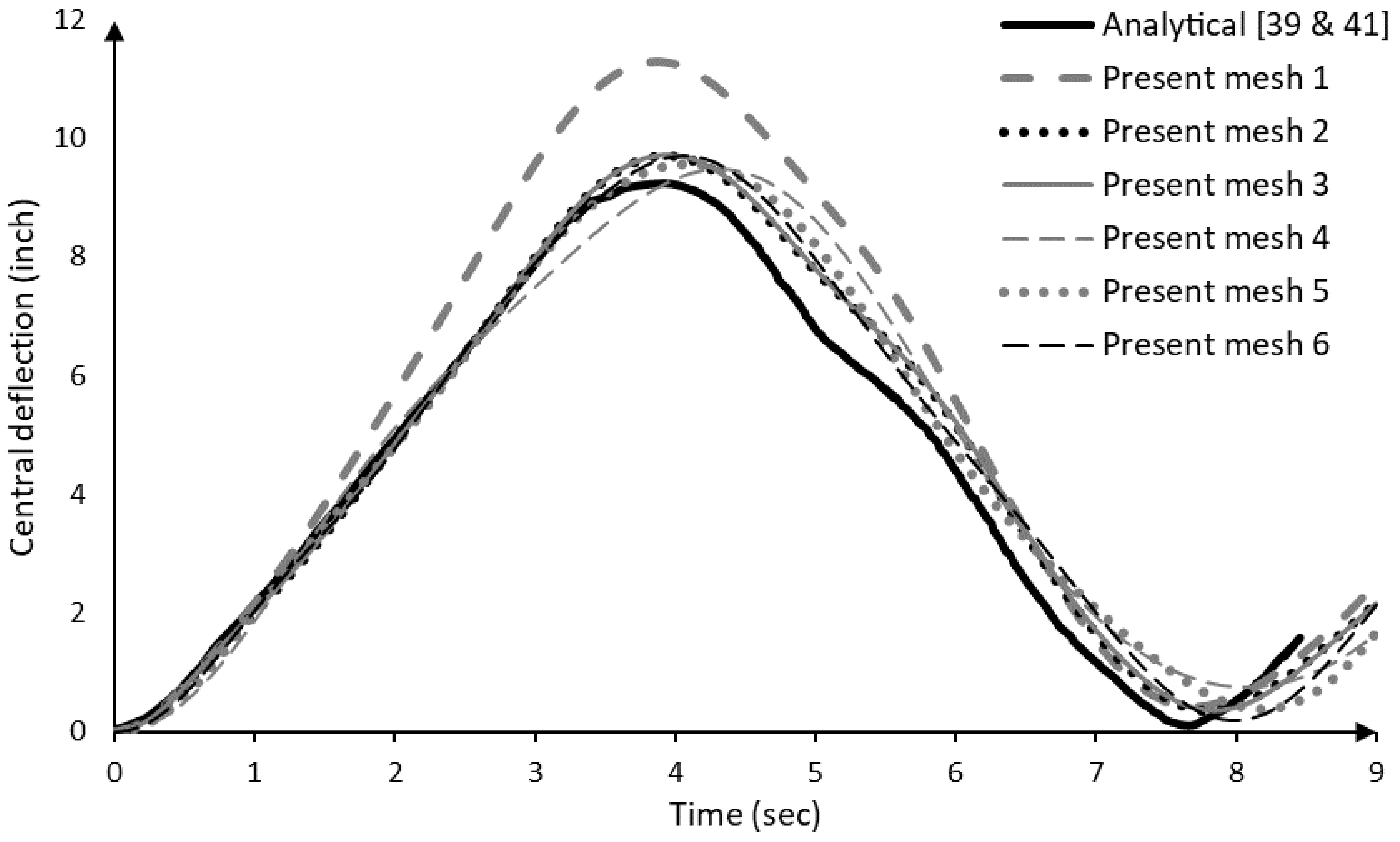
Present formulation mesh 4 (partial discretization)



Present formulation mesh 5 (partial discretization)



Present formulation mesh 6 (partial discretization)

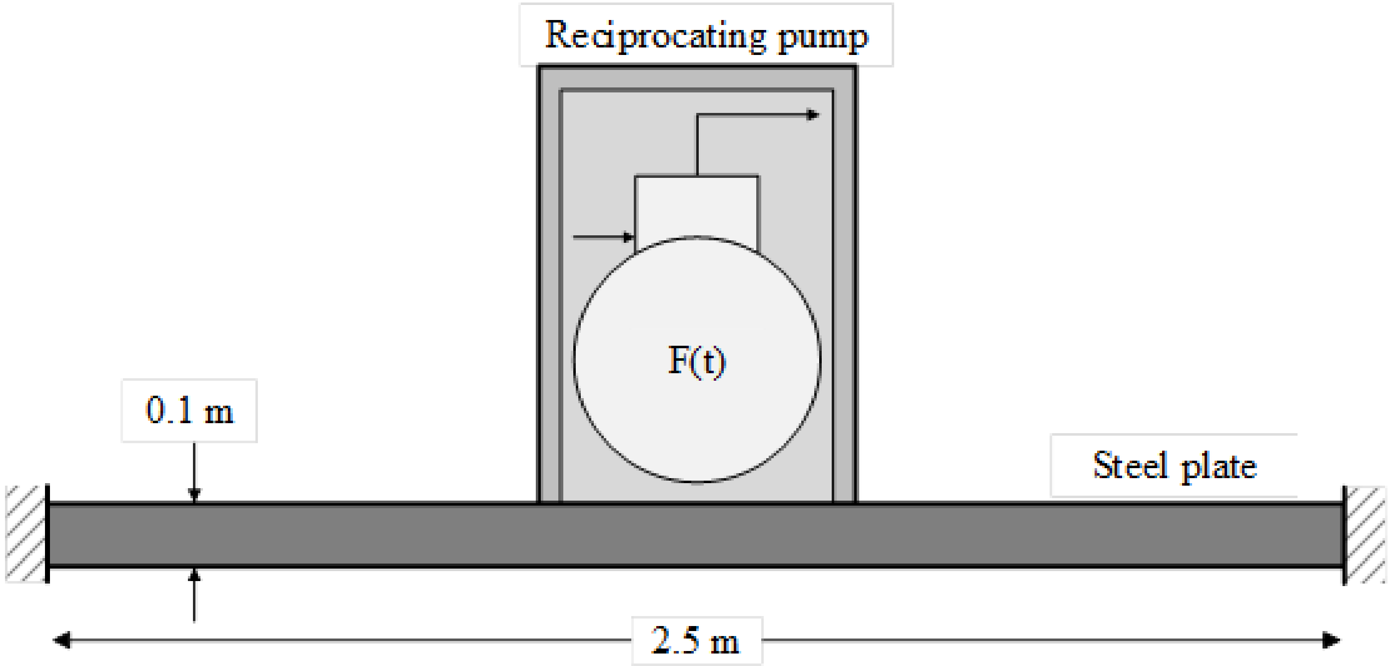


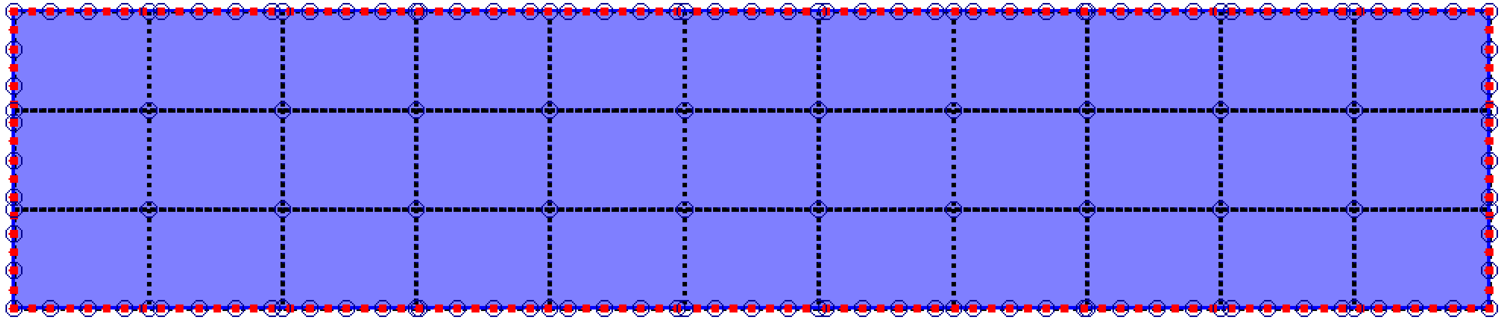
Analytical [39 & 41]
Present mesh 1
Present mesh 2
Present mesh 3
Present mesh 4
Present mesh 5
Present mesh 6
Central deflection (inch)
Time (sec)
0
2
4
6
8
10
12
0
1
2
3
4
5
6
7
8
9

Numerical Examples

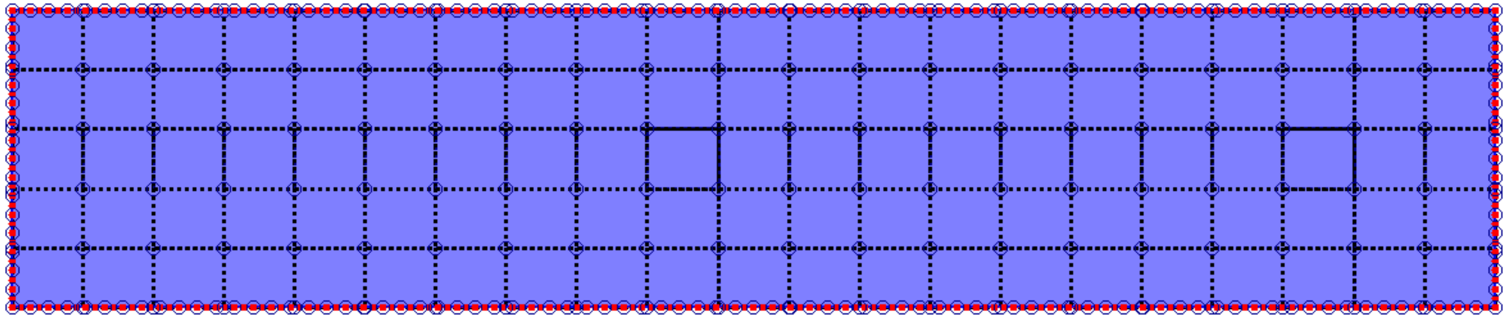
- **Example.5 plate supporting a pump**

A reciprocating pump is mounted at the middle of a steel plate clamped along two edges.
Dimensions 2.5 ×0.5 × 0.1 m.
$E = 2 \times 10^{11}\ N/m^2$.
υ = 0.3.
$\rho = 7700\ kg/m^3$.
Harmonic force, F(t) = 220 sin(62.832 t) N.
$\Delta t = 1.5625 \times 10^{-4}\ sec$.

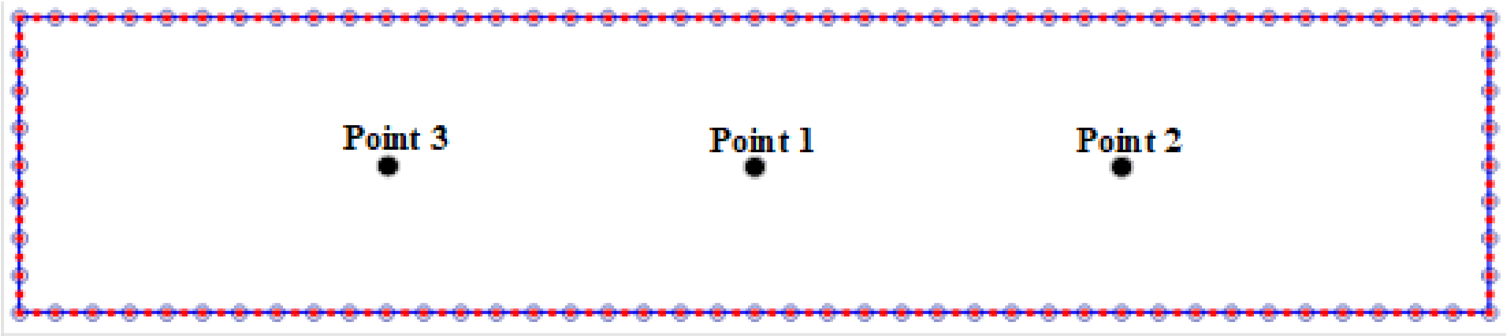




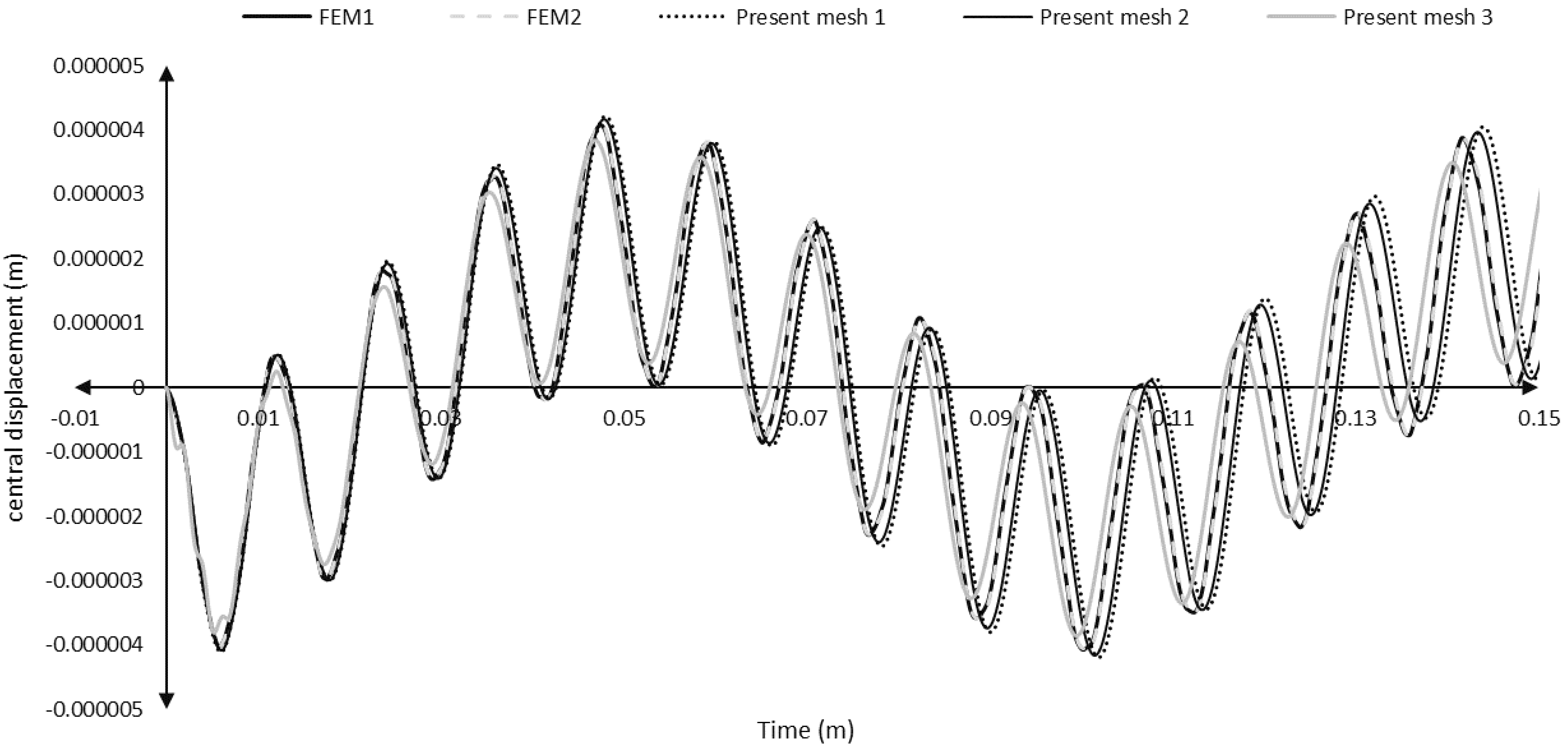
Present formulation mesh 1 (Full domain discretization)



Present formulation mesh 2 (Full domain discretization)



Present formulation mesh 3 (Full domain discretization)



FEM1
FEM2
Present mesh 1
Present mesh 2
Present mesh 3
0.000005
0.000004
0.000003
0.000002
0.000001
0
-0.000001
-0.000002
-0.000003
-0.000004
-0.000005
-0.01
0.01
0.03
0.05
0.07
0.09
0.11
0.13
0.15
central displacement (m)
Time (m)

Numerical Examples

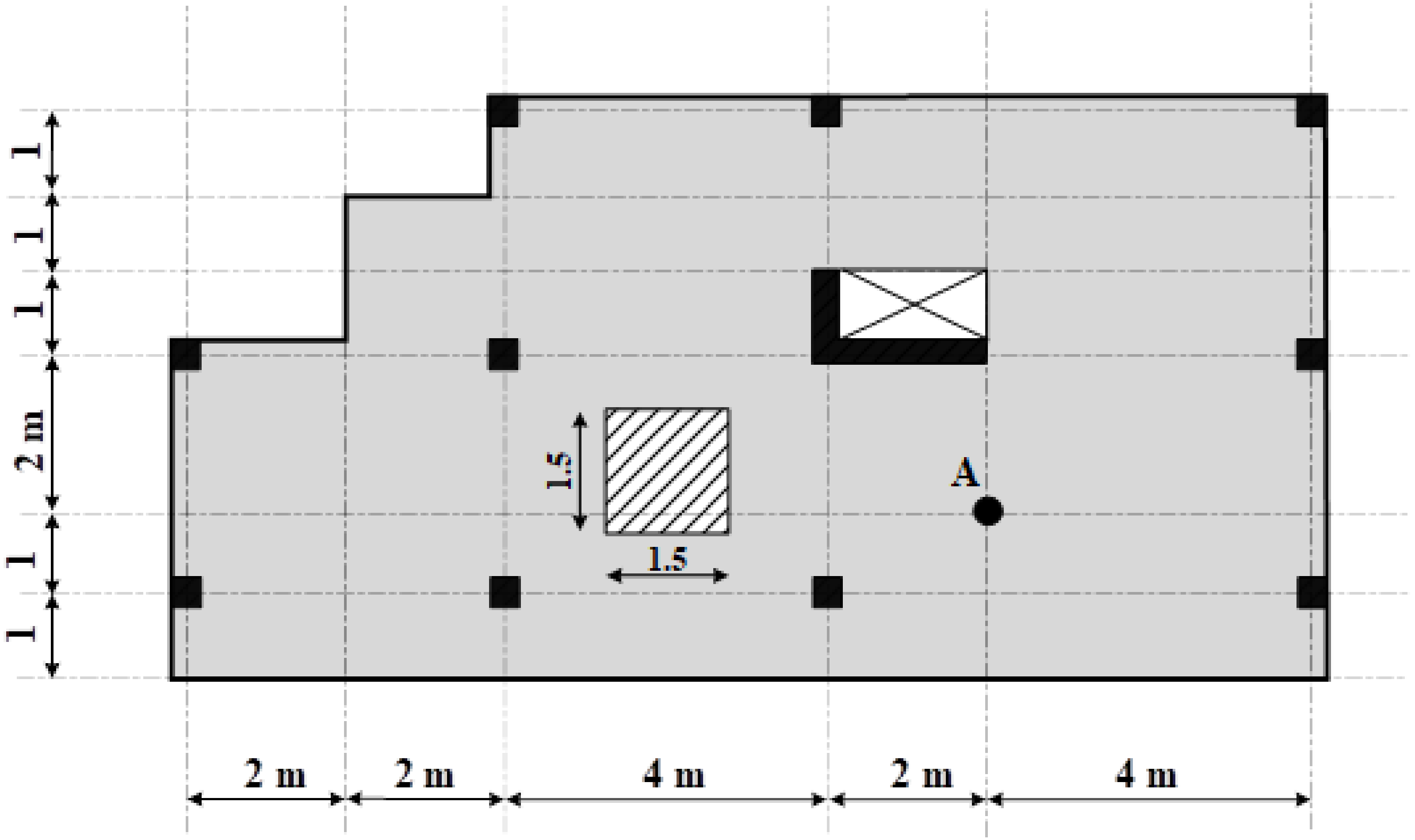
- **Example.6 Free and forced vibration of a building slab**

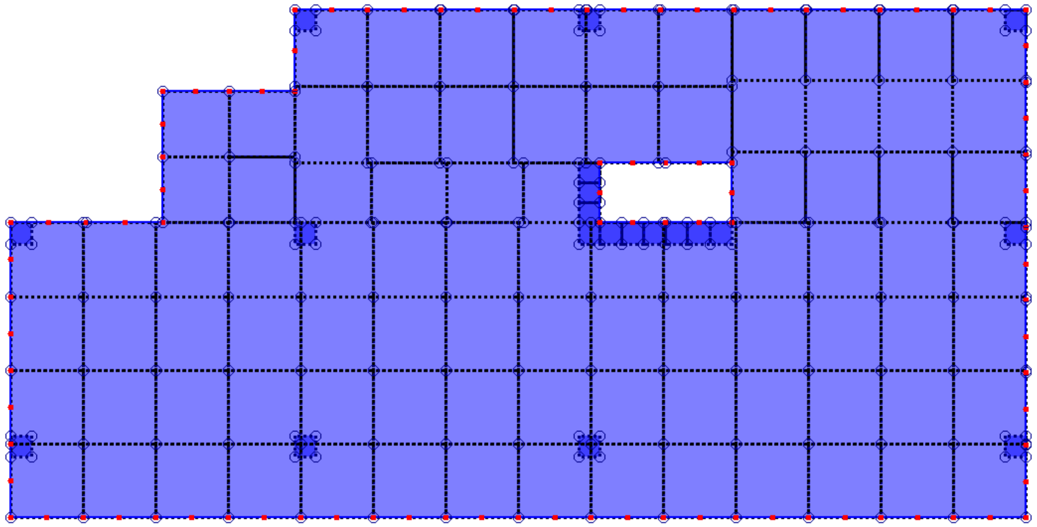
A practical slab with thickness = 0.2 m and supported on columns with dimensions (0.3×0.3) m as well as shear walls of dimensions (1.15×0.3) m, (1.85×0.3) m.

$E=2\times 10^{6}\ t/m^{2}$

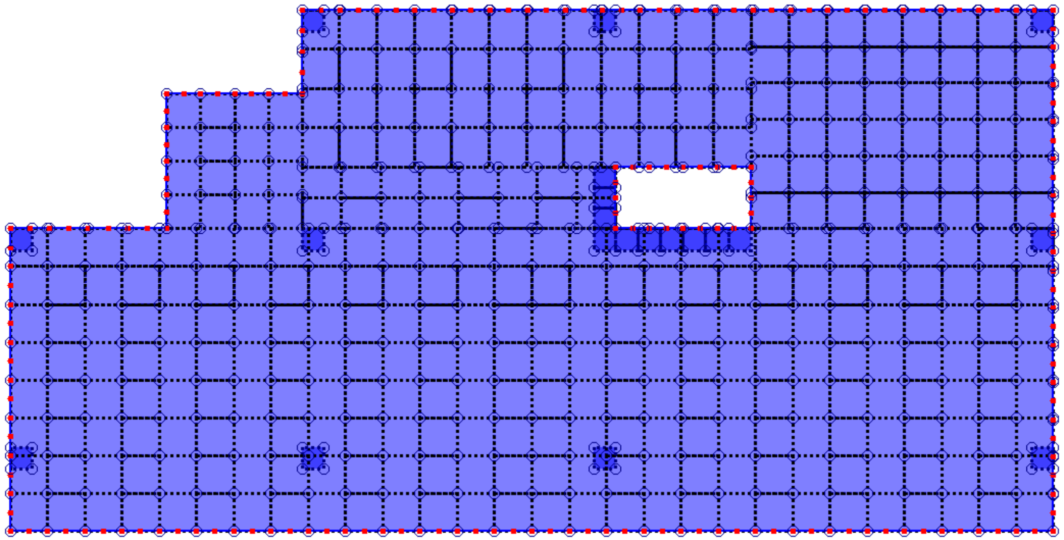
$\upsilon = 0.2$

$\rho=2.5$

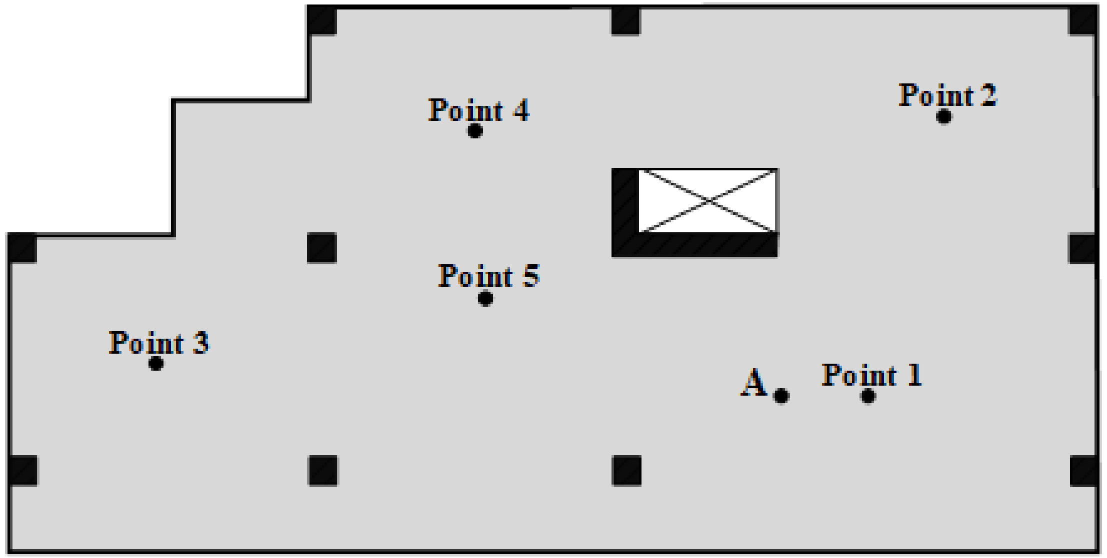




Present formulation mesh 1
(Full domain discretization)



Present formulation mesh 2
(Full domain discretization)



The used points in the partial discretization

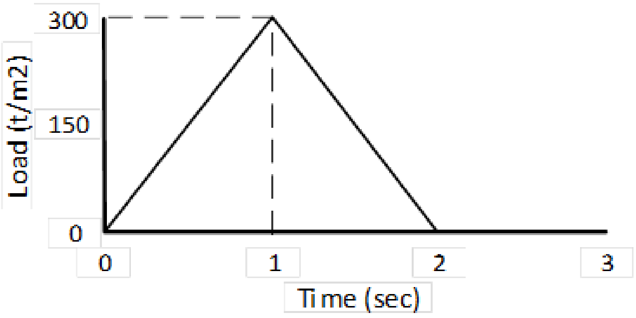
Case A: Free vibration

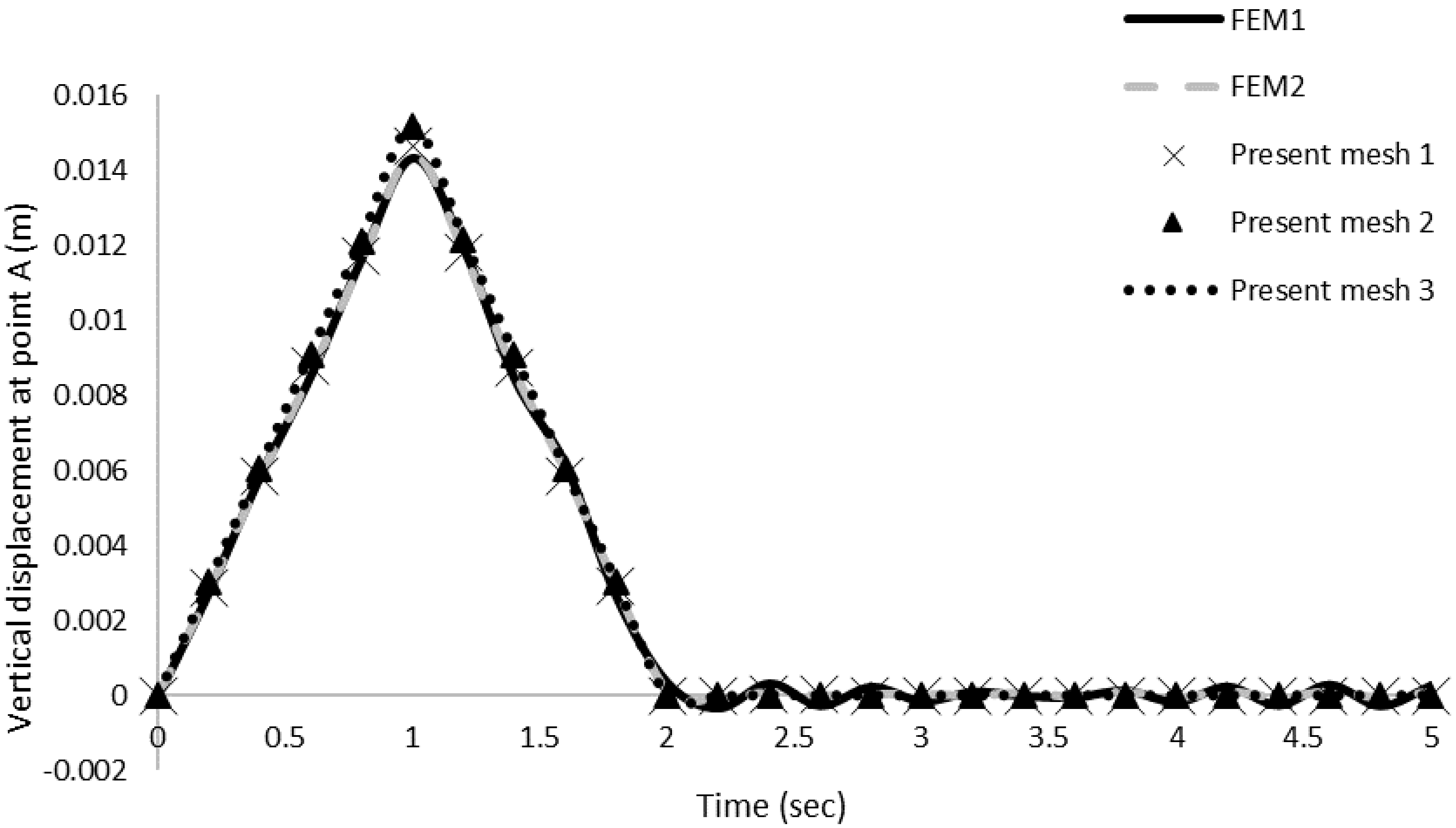
Table 6.1 Frequencies of practical slab in example 6.5, case A.

Method	Mesh	Frequencies		
		1st mode	**2nd mode**	**3rd mode**
FEM-thick plate	1131 DOFs	10.7775	11.3986	16.3449
	2268 DOFs	10.6599	11.1946	16.3156
	5868 DOFs	11.2047	12.3105	17.5553
The present formulation (Full domain discretization)	108 DOFs - 42 BEs	11.1619	12.2723	16.4481
	372 DOFs - 84 BEs	11.0625	12.1325	16.3161
The present formulation (Partial discretization)	(20 Col. + 1 point) DOFs - 42 BEs	11.3567	45.7340	46.5292
	(20 Col. + 2 point) DOFs - 42 BEs	11.3244	12.4720	45.4775
	(20 Col. + 3 point) DOFs - 84 BEs	11.3731	12.4588	18.0523
	(20 Col. + 4 point) DOFs - 84 BEs	11.3604	12.1927	16.8416
	(20 Col. + 5 point) DOFs - 84 BEs	11.2448	12.1919	16.3331

Case B: Forced vibration

The slab is subjected to the dynamic load applied on the hatched area of dimensions (1.5m×1.5m). The used time step is 0.2 second.





Conclusions

- The proposed formulation is well-suited for analyzing both free and forced vibrations.
- In this formulation, it is not essential to discretize the entire domain into cells; instead, it is sufficient to use a limited number of points.
- The results obtained from the present formulation are in good agreement with the analytical solutions compared with the results of FEM, whether plate FEM or 3D FEM.
- It requires less computational effort while providing higher accuracy than other numerical methods, as accurate results can be achieved without the need for a large number of DOFs or BEs.
- This formulation is suitable for practical slab structures, offering efficient and accurate analysis without requiring huge computational efforts or a high number of DOFs or BEs.

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Recommendations for Future work

- Dynamic analysis of post-tensioned slabs.
- Walking vibration analysis.
- Automated selection of representative mass points.
- Dynamic analysis of functionally graded and laminated thick plates.
- Analysis of soil–structure interaction and fluid–structure interaction problems.
- Incorporation of damping effects into the dynamic BEM framework.
- Application in aviation engineering.

Thank you for your attention

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