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FRACTIONAL DERIVATIVE IN MECHANICS AND ITS FUTURE PERSPECTIVES

NUMERICAL SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS

Prof. J.T. Katsikadelis, Dr. Eng., PhD, Dr. h.c., Academician

Lecture presented at the workshop
Application of BEM in cultural heritage and structural engineering industry
Organized by Prof. Youssef Rashed
Cairo University
July 13-15, 2021

AN ASPECT OF MATHEMATICS

"The formulation of physical problems reflects the mathematical tools available at the time of their development.

We should visualize mathematics as a set of mental entities that has no bounds. Thus, whenever we come across a mathematical problem resulting from the modeling of a physical system and the available mathematics are not adequate to solve this problem, we should create new mathematics to cope with it, instead of simplifying the model of the physical system, so that it is amenable to available mathematics."

(Katsikadelis 2017)

I.Modeling of Physical systems

The invention of the differential calculus and the physical laws enabled the modeling of the physical systems via differential equations at the first instance and then by integral and integrodifferential equations whose solution could predict their response.

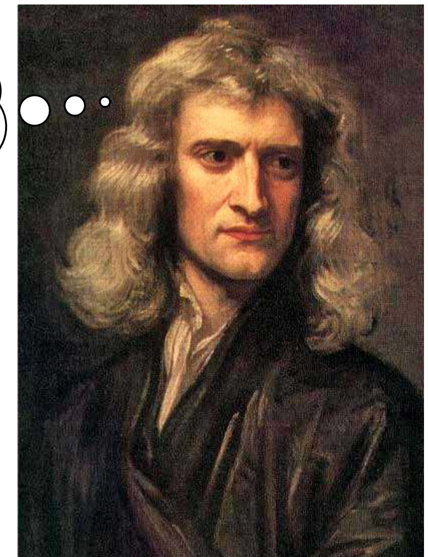
G.W. Leibniz
(1646–1716)

$$\frac{d^n y}{dx^n}$$

This actually started with the definition of the derivative and its rules by Leibniz (1675) and the statement of the laws of motion by Newton (July 5, 1686).

$$\frac{d}{dt}(mv) = f$$

We remind that Newton stated his laws as axioms (Axiomata sive Leges Motus, Principia Mathematica) without saying anything about how he concluded to them.



Sir Isaac Newton
1643 - 1727

Cairo University

J.T. Katsikadelis. "The Fractional Derivative and is Application to Physical Systems
Lecture presented at the workshop organized by Prof. Youssef Rashed, University of Cairo.

In some recent publications, it has been shown that these laws can be derived directly either from Galileo's experimental data or from Kepler's laws of planetary motion (Katsikadelis, 2015,2018 & 2019)

Springer *Derivation of Newton's law of motion using Galileo's experimental data*

John T. Katsikadelis

Acta Mech (2015) 226:3195-3204

Springer *Derivation of Newton's law of motion from Kepler's laws of planetary motion*

John T. Katsikadelis

Arch Appl Mech (2018) 88:27-38

Springer *Is Newton's law of motion really of integer differential form?*

John T. Katsikadelis

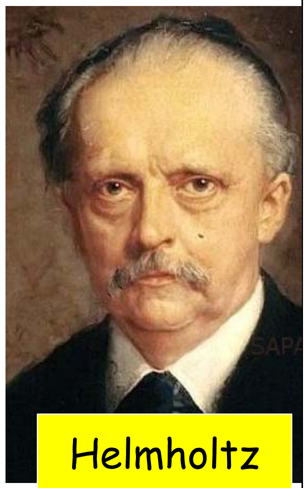
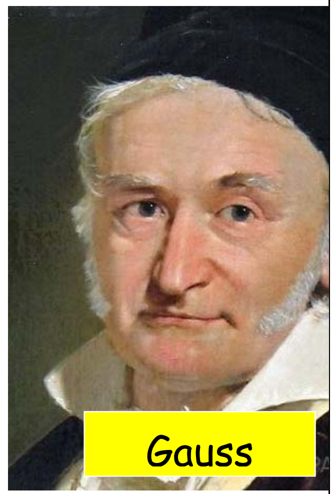
Arch Appl Mech (2019) 89:639-647

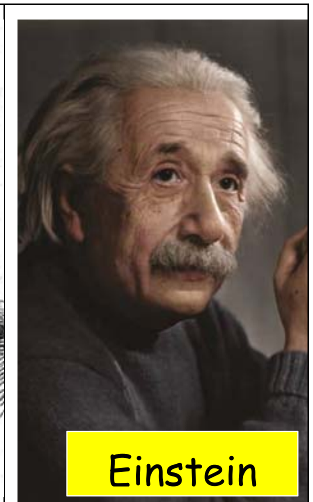
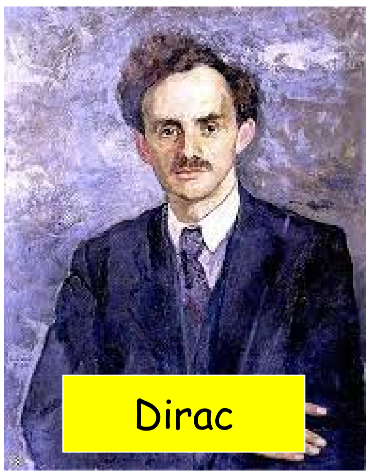
In the last paper is shown that Newton's law of motion really is of integer order differential form, a fact that is very important for the validity of the modeling of physical systems based on it.

Using the derivative and the physical laws (Newton's law, constitutive equations) many differential equations have been derived which describe the response of a system in Physics and Engineering.

Many famous scientists and mathematicians dedicated their efforts to derive such equations. I mention a few of them, which, of course, are known to all of us.


Cauchy
Navier
Poisson
Euler
Saint Venant
Fermat
Bernoulli
Kirchhoff
Sofie Germain
Lagrange


Laplace

Helmholtz

Gauss

Love

Einstein

Dirac

I. The Fractional Derivative

A.1 The fractional derivative and its definitions (Born in 1695)

Fractional (non-integer order) Derivatives are as old as Calculus.

Theory of derivatives of non-integer order goes back to G. W. Leibnitz.

After Leibnitz defined the derivative of integer order,

$\frac{\partial^n y}{\partial x^n}$, n: integer

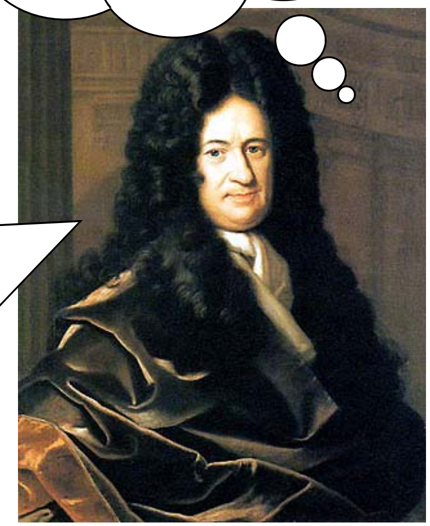
G.F.A. de L'Hôpital
(1661–1704)

L' Hôpital asked:

"What if n is a fraction, say $n = 1/2$?"

Leibnitz answered (30 September 1695) and concluded:

"*Ainsi il s'en suit que* $d^{1:2}x$ *sera égal à* $x \cdot \sqrt[2]{dx : x}$" and added prophetically "*Il y a de l'apparence qu'on tirera un jour des conséquences bien utiles de ces paradoxes, car il n'y a guerres de paradoxes sans utilité*"



G.W. Leibniz
(1646–1716)

Regarding the meaningful role of FC in modern mechanics

"It would not be excessive to say that simulating physical systems using only integer-order derivatives is similar to doing arithmetic (algebra) using only integer numbers".

(J.T. Katsikadelis, Archive of Applied Mechanics, 2015, pp. 1307-1320)

- For 3 centuries the fractional derivative inspired pure theoretical mathematical developments useful only for mathematicians.
- In the last three decays, however, many authors pointed out that fractional derivatives are very suitable for the description of real materials and physical or social procedures. Thus, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes as well nonlocal response (fractional elasticity).

Thus, several research topics gave a boost to revisiting Fractional Calculus for modeling the actual systems and the development of efficient numerical methods to solve the differential equations.

There are several definitions of the Fractional Derivative:

Almost all are of integro-differential form expressed by convolution integrals

Riemann-Liouville fractional derivative,

Grünwald-Letnikov fractional derivative,

Caputo Fractional derivative

Riemann-Liouville Definition

G.F.B. Riemann

(1826–1866)

$$ {}_aD^{\alpha}u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t \frac{u(\tau)d\tau}{(t-\tau)^{\alpha-m+1}} $$

$$ m-1 \le \alpha \le m $$

$$ {}_aD^{\alpha}u(t) = u^{(m)} \qquad \alpha = m $$

$$ {}_aD^{\alpha}u(t) = u^{(m-1)} \qquad \alpha \to (m-1)^{+} $$

J. Liouville

(1809–1882)

$\alpha :=$ the order of the fractional derivative

$m :=$ integer $\quad a :=$ lower bound

$\Gamma(z) :=$ The Gamma function

$$ \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad z \in C \text{ complex number} $$

$$ \Gamma(1) = \int_0^{\infty} e^{-t} dt = 1 \qquad \Gamma(z+1) = z\Gamma(z) \qquad \Gamma(n) = n! = 1 \cdot 2 \cdot 3, \ldots, n \quad \text{the factorial} $$

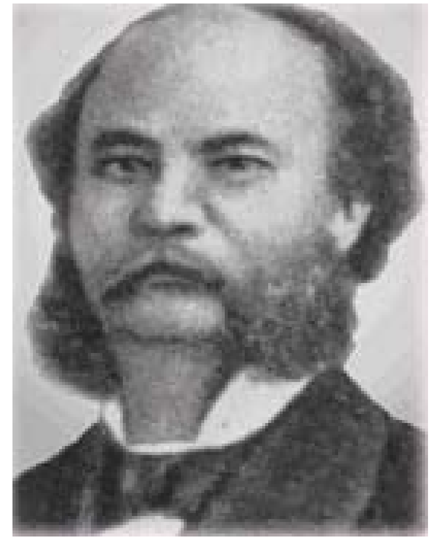
A.K. Grünwald
1838-1920

Grünwald - Letnikov definition

$$ {}_aD^{\alpha}u(t) = \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} u(t-kh) $$

$[x]$ − integer part of x

$$ {}_aD^{\alpha}_{G-Letn}\, u(t) \rightleftarrows D^{\alpha}_{R-Liouv} $$



A.V. Letnikov
1837-1888

Laplace transform of the R-L Derivative

$$ L\{{}_0D^{\alpha}u(t)\} = s^{\alpha}U(s) - \sum_{k=0}^{m-1} s^k \left[{}_0D^{\alpha-k-1}u(t)\right]_{t=0} $$

$\left[{}_0D^{\alpha-k-1}u(t)\right]_{t=0} =$ has no direct physical meaning It does not allow to apply initial conditions with physical meaning, i.e. $u(0), u'(0), u''(0), \ldots$

Caputo definition (1967)

A solution to this conflict was proposed by M. Caputo in 1967, who defined a fractional derivative allowing the application of initial conditions with physical meaning

Michele Caputo

$$D_C^{\alpha}u(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{u^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau$$

$$\lim_{\alpha\to m}D_C^{\alpha}u(t)=u^{(m)}(t) \qquad \lim_{\alpha\to m-1}D_C^{\alpha}u(t)=u^{(m-1)}(t)-u^{(m-1)}(0)$$

$$D_{RL}^{\alpha}u(t)\equiv D_C^{\alpha}u(t) \quad m-1<\alpha<m \quad \text{if } u^{(m-1)}(0)=0$$

Laplace transform of the Caputo derivative

$$L\left[D_c^{\alpha}u(t)\right]=s^{\alpha}U(s)-\sum_{k=0}^{m-1}s^{\alpha-k-1}u^{(k)}(0), \qquad m-1<\alpha<m$$

The Caputo Derivative is suitable to apply initial conditions with physical meaning, even the FD does not have geometrical meaning

What is the physical meaning of the FD? Long discussion (I. Podlubny, 2002)

But this shortcoming of the FD had major implications in the development of Science:

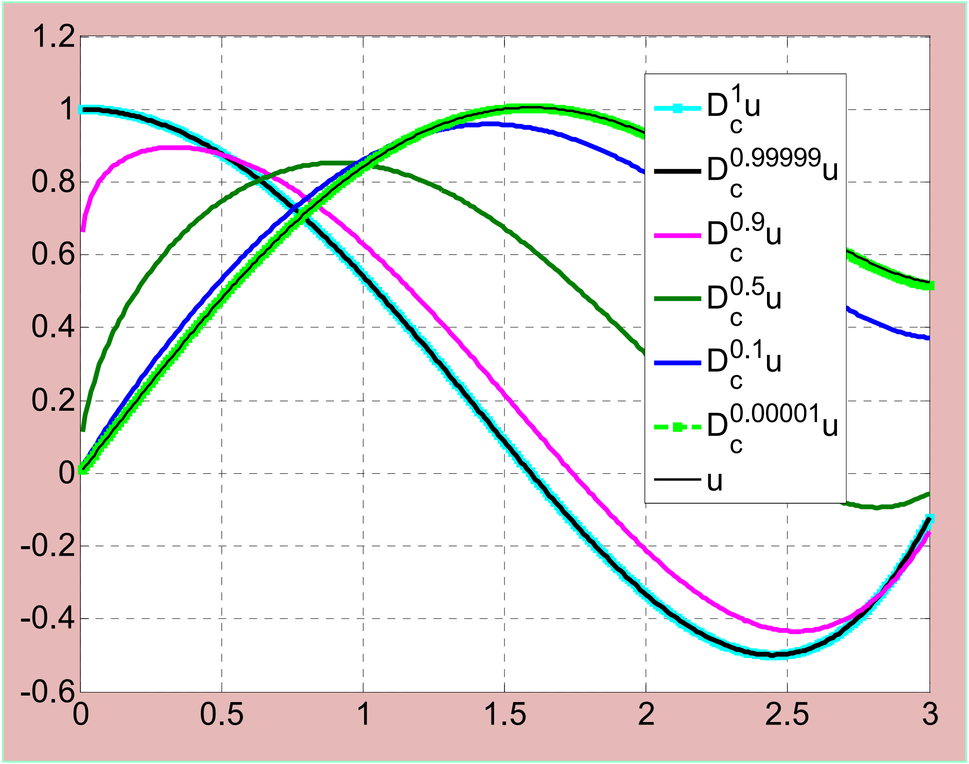
Has Newton's law of motion delayed the development of Science?

The integer order derivative allows giving geometrical interpretations to the proposed physical models resulting from Newton's laws. Apparently, this made the then revolutionary concepts accessible to the contemporary scientists, who were well experienced in geometry.

In a sense, the long delay to apply Fractional Calculus may be attributed to this fact.

We recall that fractional derivatives lack the straightforward geometrical interpretation of their integer counterparts.

Graphical representation of the FD of a function



$$u = t - \frac{1}{6}t^3 + \frac{1}{120}t^5$$

$$\frac{du}{dt} = 1 - \frac{3}{6}t^2 + \frac{5}{120}t^4$$

$$D_c^{0.5}u(t) = 1.13t^{1/2} - .301t^{5/2} + .0191t^{9/2}$$

The generalized Fractional Derivative (Katsikadelis' definition 2013)

8th German-Greek-Polish Symposium

Goslar, Germany (2013)

Generalized fractional derivatives and beyond them. Applications to mechanical systems

John t. Katsikadelis

Springer

Archive of Applied Mechanics

Vol.85 pp.1307-13-20 1320 (2014) on line (2014)(

Generalized Fractional Derivatives and their Applications to Mechanical Systems

John T. Katsikadelis

$$D^{\alpha}u(t) = \begin{cases} \alpha \to m & \to u^{(m)} \\ \alpha \to m-1 & \to u^{(m-1)} \end{cases}$$

The FD: interpolation between successive integer order derivatives

This property is used to define new FDs, (GFDs), by the convolution integral

$$D_G^{\alpha}u(t) = \begin{cases} \int_0^t f(t-\tau;\alpha,m)u^{(m)}(\tau)d\tau & m-1<\alpha<m \\ u^{(m)}(t) & \alpha = m \end{cases}$$

Is such a definition Possible?

Yes, It is possible if the kernel $f(t;\alpha,m)$ has the following two properties:

P1: $\underset{\alpha \to m-1}{f(t;\alpha,m)} = 1$ P2: $\underset{\alpha \to m}{L[f(t;\alpha,m)]} = 1$

These two properties hold

Question:

Can we find (construct) kernel functions that have properties P1 and P2 ?

Answer: YES

TABLE 1 Families of kernels producing generalized FDs **(**$m-1<\alpha<m$**)**

	$f(t;\alpha,m)$, ($\theta(\alpha)>0$, any specified function of α)	$L[f(t;\alpha,m)]$
1	$\dfrac{1}{\Gamma\left((m-\alpha-1)\dfrac{\theta(\alpha)}{\theta(m)}+1\right)}\dfrac{1}{t^{(\alpha+1-m)\frac{\theta(\alpha)}{\theta(m)}}}$	$s^{(\alpha-m+1)\frac{\theta(\alpha)}{\theta(m)}-1}$
2	$\dfrac{1}{\Gamma\left((m-\alpha-1)\dfrac{\theta(\alpha)}{\theta(m)}+1\right)}\dfrac{1}{[\exp(t)-1]^{(\alpha+1-m)\frac{\theta(\alpha)}{\theta(m)}}}$	$\dfrac{\Gamma\left(s+(-m+\alpha+1)\dfrac{\theta(\alpha)}{\theta(m)}\right)}{\Gamma(s+1)}$

Obviously, both families satisfy the properties

$$\underset{\alpha\to m-1}{f(t;\alpha,m)}=1 \qquad \underset{\alpha\to m}{L[f(t;\alpha,m)]}=1$$

This definition generates infinite classes of FDs.

Examples for family#1 $f(t;\alpha,m)=\dfrac{1}{\Gamma\left((m-\alpha-1)\dfrac{\theta(\alpha)}{\theta(m)}+1\right)}\dfrac{1}{t^{(\alpha+1-m)\frac{\theta(\alpha)}{\theta(m)}}}$

Choose
$$\theta(\alpha)=c_0+c_1\alpha+c_2\alpha^2+\ldots+c_k\alpha^k$$
$$\theta(\alpha)=\exp(c_0+c_1\alpha+c_2\alpha^2+\cdots c_k\alpha^k)$$
$$\theta(\alpha)=\sin(\alpha)$$

For $0<\alpha<1 \quad (m=1)$

$\theta(\alpha)=1$	$D^{\alpha}_{G1}u(t)=\dfrac{1}{\Gamma(1-\alpha)}\int_0^t\dfrac{u^{(1)}(\tau)}{(t-\tau)^{\alpha}}d\tau$ (Caputo FD)
$\theta(\alpha)=1+2\alpha$	$D^{\alpha}_{G2}u(t)=\dfrac{1}{\Gamma(1-(\alpha+2\alpha^2)/3)}\int_0^t\dfrac{u^{(1)}(\tau)}{(t-\tau)^{(\alpha+2\alpha^2)/3}}d\tau$
$\theta(\alpha)=\sin(\alpha)$	$D^{\alpha}_{G3}u(t)=\dfrac{1}{\Gamma(1-\alpha\sin(\alpha)/\sin(1))}\int_0^t\dfrac{u^{(1)}(\tau)}{(t-\tau)^{\alpha\sin(\alpha)/\sin(1)}}d\tau$

Let $u(t) = t^2$. Then	
$D^{\alpha}_{G1}u(t) = \dfrac{2t^{2-a}}{\Gamma(3-\alpha)}$ $D^{\alpha}_{G2}u(t) = \dfrac{2t^{2-(\alpha+2\alpha^2)/3}}{\Gamma[3-(\alpha+2\alpha^2)/3]}$	$D^{a}_{G3}u(t) = \dfrac{2t^{2.-0.01832a\exp(1.+3a^2)}}{\Gamma[3-0.01832a\exp(1+3a^2)]}$

What is the usefulness of the Generalized Fractional Derivative?

one parameter → $D^{\alpha}u(t)$

any number of parameters → $D_G^{\alpha;c_1,c_2,\ldots,c_3}u(t)$

At first sight the new FDs provide, beside the order α of the FD, any number of parameters to better calibrate a physical response. Their further usefulness and consequences is subject of investigation

Applications of the Fractional Derivative

Wide application of fractional Calculus in various disciplines the last 3 decays, which is rapidly increasing

- Bioengineering (modeling human)
- Signal processing
- Social sciences
- Modeling of dissipative forces
- Modeling materials with hereditary properties (viscoelasticity)

The description of physical laws (constitutive equations) via fractional derivatives leads to differential equations involving derivatives of noninteger order

II Fractional Differential Equations
a. Constant Order

a. Ordinary fractional differential equations

$$a_1 D_c^{\alpha_1} u(t) + a_2 D_c^{\alpha_2} u(t) + a_3 D_c^{\alpha_3} u(t) + \cdots + a_n D_c^{\alpha_n} u(t) = p(t), \quad (1)$$
$$a_i = a_i(t),\ a_1 \neq 0, \quad 0 \le \alpha_n < \alpha_{n-1} < \cdots < \alpha_1$$

Examples:

Fractional oscillator:

$$m\ddot{u} + cD_c^{\alpha} u + ku = p(t), \quad u(0) = u_0,\ \dot{u}(0) = \dot{u}_0 \ ,$$

Fractional Duffing oscillator:

$$m\ddot{u} + cD_c^{\alpha} u + k_1 u + k_2 u^2 = p(t), \quad u(0) = u_0,\ \dot{u}(0) = \dot{u}_0 \ ,$$

a. Partial fractional differential equations

$$N(u)+\rho D_c^{\beta}u+\eta D_c^{\alpha}u=g(\mathbf{x},t), \quad \mathbf{x}\in\Omega, \quad t>0, \quad 0<\alpha\leq 1, \quad 1<\beta\leq 2 \qquad (1)$$

$$B(u)=\bar{g}(\mathbf{x},t), \quad \mathbf{x}\in\partial\Omega \qquad (2)$$

$$u(\mathbf{x},0)=f_1(\mathbf{x}), \quad \dot{u}(\mathbf{x},0)=f_2(\mathbf{x}), \quad \mathbf{x}\in\Omega,\ t>0 \qquad (3)$$

$N(\cdot), B(\cdot)$: Linear or nonlinear integer order partial differential operators with respect to x, y.

$D_c^{\alpha}u(t)$, $D_c^{\beta}u(t)$: *Generalized fractional time derivative of* α - and - β, which represent fractional type damping and inertia forces respectively;

3. Solution of FDEs

The theory and analysis of FDEs has been well established. Theorems of existence, uniqueness and behavior of the solution of ordinary and partial FDEs have been thoroughly investigated by the mathematicians. All this on a theoretical background.

Excellent books have been published, e.g.,

On fractional Calculus (Oldham & Spanier 1974, Carpinteri & Mainardi (Eds) 1997)

On fractional Differential Equations (Miller & Ross 1993; Podlubny 1999; Kilbas et al. 2006).

1. Analytic solutions:

Due to mathematical complexity the to date solutions are very few and are restricted to one dimensional domains, or e.g.

- Atanackovic, T.M. (2002),
- Atanackovic, T.M., Stankovic, B. (2002)
- Katica (Stepanovic) Hendrieh, (2006)

2. Approximate solutions

1. Rossikhin, Yu. A. Shitikova, M.V. (2006a), (2006b)

3. Numerical solutions:

3a. The FEM (Finite Element Method); could be possibly (?) employed. There are few available solutions for linear problems in the literature. e.g. FEM based solution combined with approximate methods to solve the semi-discretized fractional evolution equations, e.g

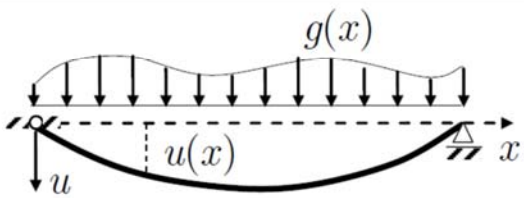
1 Gaul, L. (1999)
2 Schmidt, A. & Gaul, L. (2002)
3 Galucio, A.C., Deu, J. –F. & Ohayon, R. (2004)

3b. The AEM (Analog Equation Method): It works as a general method to solve linear and nolinear integer order or fractional order PDEs. This method is based on the Principle of the Analogue Equation

The Principle of the Analog Equations

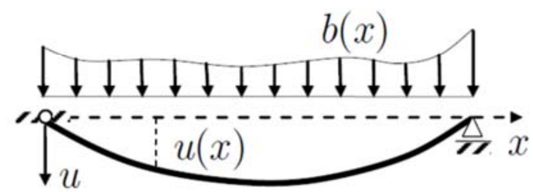
(Katsikadelis 1994)

Real Problem	Substitute Problem
$N(u)=g(\mathbf{x}), \quad \mathbf{x}\in\Omega$	$L(u)=b(\mathbf{x}), \quad \mathbf{x}\in\Omega$
$B(u)=\bar{g}(\mathbf{x}), \quad \mathbf{x}\in\Gamma$	$B(u)=\bar{g}(\mathbf{x}), \quad \mathbf{x}\in\Gamma$
$N(\ )$: Linear or nonlinear operator	$L(\ )$: Linear operator of the order of $N(\ )$



$$[EI(x)u(x)_{,xx}]_{,xx}=g(x)$$

(a)



$$u(x)_{,xxxx}=b(x)$$

(b)

J.T. Katsikadelis. "The Fractional Derivative and is Application to Physical Systems
Lecture presented at the workshop organized by Prof. Youssef Rashed, University of Cairo.

$$N(u)+\rho D_c^{\beta}u+\eta D_c^{\alpha}u=g(\mathbf{x},t)\,.,\qquad \mathbf{x}\in\Omega,\quad t>0\,,\quad 0<\alpha\le 1,\quad 1<\beta\le 2 \qquad (1)$$

$$B(u)=\bar{g}(\mathbf{x},t)\,,\quad \mathbf{x}\in\partial\Omega \qquad (2)$$

$$u(\mathbf{x},0)=f_1(\mathbf{x})\,,\qquad \dot{u}(\mathbf{x},0)=f_2(\mathbf{x})\,,\qquad \mathbf{x}\in\Omega,\ t>0 \qquad (3)$$

Using the Principle of the Analog Equation the IBVP (1), (2), (3) is converted into the substitute problem

$$L(u)=b(\mathbf{x},t)\,,\quad \mathbf{x}\in\Omega,\ t>0 \qquad (5)$$

$$B(u)=\bar{g}(\mathbf{x},t)\,,\quad \mathbf{x}\in\partial\Omega \qquad (2)$$

$$u(\mathbf{x},0)=f_1(\mathbf{x})\,,\qquad \dot{u}(\mathbf{x},0)=f_2(\mathbf{x})\,,\qquad \mathbf{x}\in\Omega,\ t>0 \qquad (3)$$

$L(\cdot)$: Linear operator with known fundamental solution

$b(\mathbf{x},t)$: Fictitious source, unknown in the first instance.

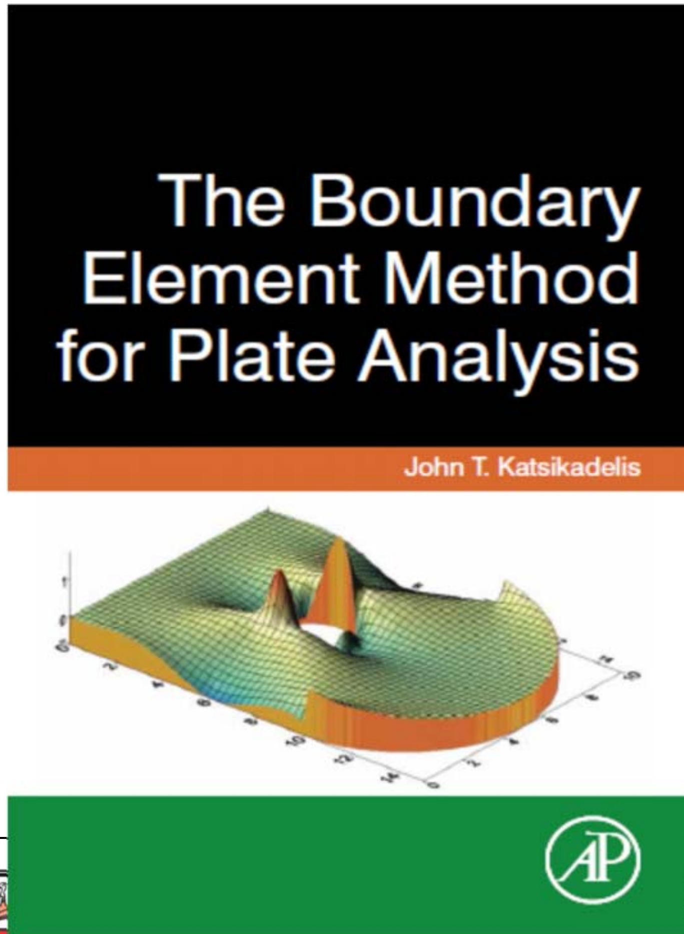
The linearity of $L(\cdot)$ and the known fundamental solution $v(\mathbf{x},\mathbf{y})$ permits the establishment of the integral representation of the solution of (5), e.g. for 4th order operator

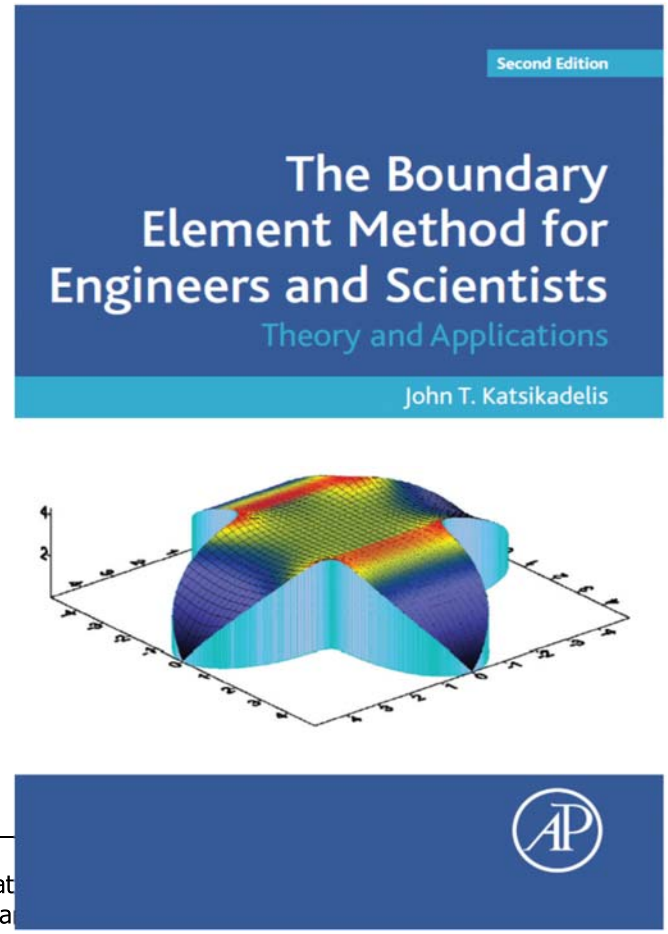
$$u=\int_{\Omega} v\,b(\mathbf{x},t)\,d\Omega-\int_{\Gamma}[vV(u)-uV(v)+v_{,n}M(u)-v_{,n}M(v)]ds\,,\quad \mathbf{x}\in\Omega,\ t>0 \qquad (6)$$

Unknown domain quantity

Unknown boundary quantities

Detailed description of the method can be found in the books by J.T. Katsikadelis





The AEM for the solution of FDEs has been presented in the papers:

ZAMM Journal of Applied Mathematics and Mechanics
Zeitschrift für Angewandte Mathematik und Mechanik

Vol. 89, No. 7, pp. 593 – 608 (2009)

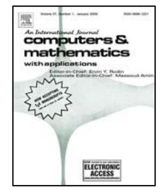
Numerical solution of multi-term fractional differential equations

John T. Katsikadelis

ELSEVIER

Computers and Mathematics with Applications

Vol.62 pp. 891–901 (2011)



The BEM for numerical solution of partial fractional differential equations

John T. Katsikadelis

The developed method for partial FDEs is general and has been employed the last years to solve problems of Mathematical Physics and Engineering described by FDEs. A list a publications using this method is given below

Katsikadelis, J.T., Babouskos, N.G. (2010), "Post-buckling Analysis of Viscoelastic Plates with Fractional Derivative Model," Engineering Analysis with Boundary Elements, **34**, 1038–1048.

Babouskos, N.G., Katsikadelis, J.T. (2010), "Nonlinear Vibrations of Viscoelastic Plates of Fractional Derivative Type: An AEM Solution," The Open Mechanics Journal 4, 8-20.

Nerantaki, M.S and Babouskos N.G. (2010), "Dynamic analysis of inhomogeneous anisotropic viscoelastic bodies described with fractional derivative models", International **Journal of Structural Stability and Dynamics**

Katsikadelis, J.T. (2009), "The fractional Diffusion-Wave Equation in Bounded Inhomogeneous Anisotropic Media", In: Recent Advances in Boundary Elements, 255-276, Springer.

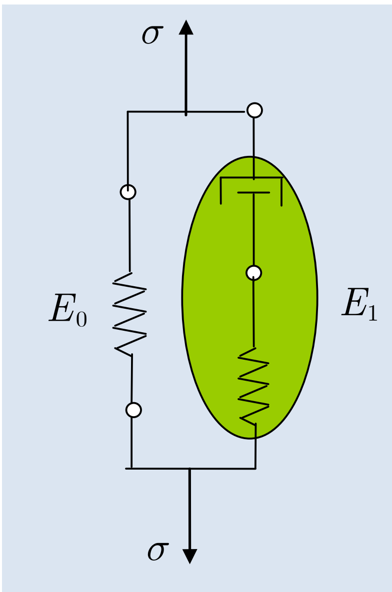
Katsikadelis J.T. (2009). "Nonlinear Vibrations of Viscoelastic Membranes of Fractional Derivative Type", Proc. BeTeq'09, July 22-24, Athens, Greece.

Katsikadelis, J.T. (2010), "Nonlinear Resonance of Viscoelastic Membranes of Fractional Derivative Type, Proc. 9th HSTAM Conference, July 12-14, Limassol, Cyprus.

Katsikadelis J.T. (2008). "Fractional Vibrations of Inhomogeneous Membranes", Proc. 6th GRACM Conference, June19-21, Thessaloniki, Greece

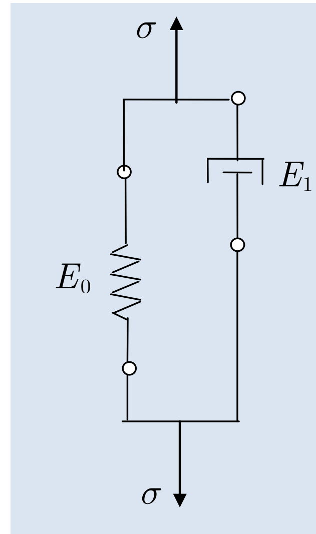
5. Viscoelastic Structures. -Fractional order differential viscoelastic models

Kelvin-Voigt model and limiting cases



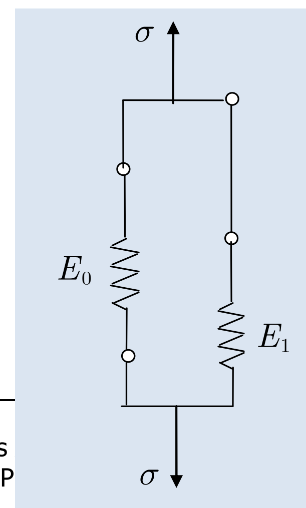
$$\sigma(t) = E_0\varepsilon(t) + E_1 D^{\alpha}\varepsilon(t)$$

$\alpha = 1$



$$\sigma(t) = E_0\varepsilon(t) + E_1 \frac{d}{dt}\varepsilon(t)$$

$\alpha = 0$

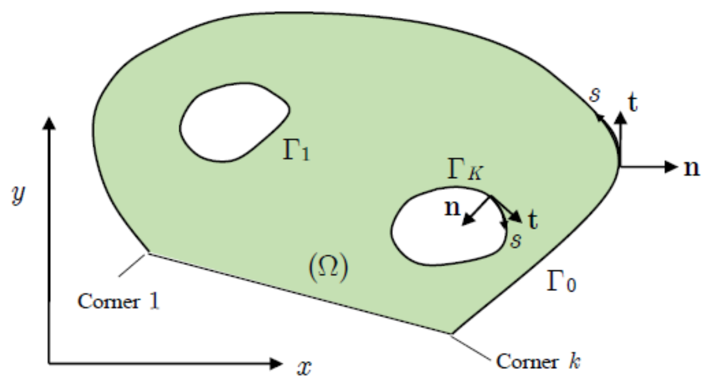


$$\sigma(t) = (E_0 + E_1 \times 1)\varepsilon(t)$$

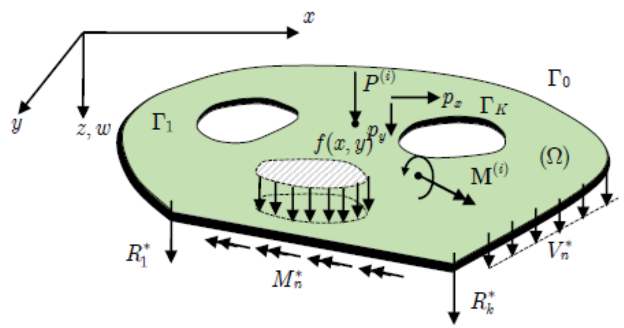
Differential Viscoelastic Models		
Viscoelastic Model	Integer derivative type	Fractional derivative type
Kelvin-Voigt	$\sigma(t) = E\varepsilon(t) + E_1 \dfrac{d\varepsilon(t)}{dt}$	$\sigma(t) = E_0\varepsilon(t) + E_1 D^{\alpha}\varepsilon(t)$
Maxwell	$\sigma(t) + b\dfrac{d\sigma(t)}{dt} = E_0\varepsilon(t)$	$\sigma(t) + bD^{\beta}\sigma(t) = E_0\varepsilon(t)$
Zener	$\sigma + b\dfrac{d\sigma}{dt} = E\varepsilon + E_1\dfrac{d\varepsilon}{dt}$	$\sigma(t) + bD^{\beta}\sigma(t) = E_0\varepsilon(t) + E_1 D^{\alpha}\varepsilon(t)$
Multi-element	$\sum_{k=0}^{n} p_k \dfrac{d^k\sigma(t)}{dt^k} = \sum_{k=0}^{m} q_k \dfrac{d^k\varepsilon(t)}{dt^k}$	$\sum_{k=0}^{n} p_k D^{\alpha k}\sigma(t) = \sum_{k=0}^{m} q_k D^{\alpha k}\varepsilon(t)$

D^{α} is the operator of the fractional derivative of order α.

6. Viscoelastic plates and membranes



Multiply connected domain Ω occupied by the plate in the xy plane bounded by the boundary $\Gamma \cup_{k=0}^{K} \Gamma_k$



Loads acting on the plate.

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The IBVP for the plate in terms of the displacements

System of three coupled partial FDEs

$$K\nabla^4 w-(N_x w_{,xx}+2N_{xy}w_{,xy}+N_y w_{,yy})+\eta K D_c^a\nabla^4 w-\eta(w_{,xx}D_c^a N_x+2w_{,xy}D_c^a N_{xy}+w_{,yy}D_c^a N_y)$$

$$+(b_x-\rho h\ddot{u})w_{,x}+(b_y-\rho h\ddot{v})w_{,y}+\rho h\ddot{w}=g(\mathbf{x},t)$$

Viscoelastic terms

Inertia terms

$$\nabla^2 u+\frac{1+\nu}{1-\nu}(u_{,x}+v_{,y})_{,x}+w_{,x}\left(\frac{2}{1-\nu}w_{,xx}+w_{,yy}\right)+\frac{1+\nu}{1-\nu}w_{,xy}\,w_{,y}$$

$$+\eta D_c^a\left\{\nabla^2 u+\frac{1+\nu}{1-\nu}(u_{,x}+v_{,y})_{,x}+w_{,x}\left(\frac{2}{1-\nu}w_{,xx}+w_{,yy}\right)+\frac{1+\nu}{1-\nu}w_{,xy}\,w_{,y}\right\}+\frac{b_x}{Gh}-\frac{\rho}{G}\ddot{u}=0$$

$$\nabla^2 v+\frac{1+\nu}{1-\nu}(u_{,x}+v_{,y})_{,y}+w_{,y}\left(\frac{2}{1-\nu}w_{,yy}+w_{,xx}\right)+\frac{1+\nu}{1-\nu}w_{,xy}\,w_{,y}$$

$$+\eta D_c^a\left\{\nabla^2 v+\frac{1+\nu}{1-\nu}(u_{,x}+v_{,y})_{,y}+w_{,y}\left(\frac{2}{1-\nu}w_{,yy}+w_{,xx}\right)+\frac{1+\nu}{1-\nu}w_{,xy}\,w_{,y}\right\}+\frac{b_y}{Gh}-\frac{\rho}{G}\ddot{v}=0$$

Nonlinear terms

Viscoelastic terms

Inertia terms

Boundary conditions on Γ

$$Vw + \eta D_c^a Vw + N_n^* w_{,n} + \eta D_c^a N_n^* w_{,n} + N_t^* w_{,t} + \eta D_c^a N_t^* w_{,t} + k_T w = V_n^* \quad \text{or} \quad w = w^* \quad \text{on } \Gamma$$

$$Mw + \eta D_c^a Mw + k_R w_{,n} = M_n^* \quad \text{or} \quad w_{,n} = w_{,n}^* \quad \text{on } \Gamma$$

$$k_T^{(k)} w^{(k)} - [\![Tw]\!]_k - \eta [\![D_c^a Tw]\!]_k = 0 \quad \text{or} \quad w^{(k)} = w_k^* \quad \text{at corner } k$$

$$N_n + \eta D_c^a N_n = N_n^* \quad \text{or} \quad u_n = u_n^* \quad \text{on } \Gamma$$

$$N_t + \eta D_c^a N_t = N_t^* \quad \text{or} \quad u_t = u_t^* \quad \text{on } \Gamma$$

Initial conditions in Ω

$$w(\mathbf{x},0) = g_1(\mathbf{x}), \quad \dot{w}(\mathbf{x},0) = g_2(\mathbf{x})$$

$$u(\mathbf{x},0) = h_1(\mathbf{x}), \quad \dot{u}(\mathbf{x},0) = h_2(\mathbf{x})$$

$$v(\mathbf{x},0) = h_1(\mathbf{x}), \quad \dot{v}(\mathbf{x},0) = h_2(\mathbf{x})$$

7. Problems resulting from the previously derived plate equations and solved using the presented method

7.1 Linear and Nonlinear Quasi-static Analysis of Viscoelastic plates

7.2 Linear and Nonlinear vibrations of viscoelastic plates

7.3 Post buckling response of viscoelastic plates

7.4Linear and nonlinear flutter instability of viscoelastic plates

7.5Non linear vibrations of viscoelastic membranes

7.5Large deflections of viscoelastic membranes

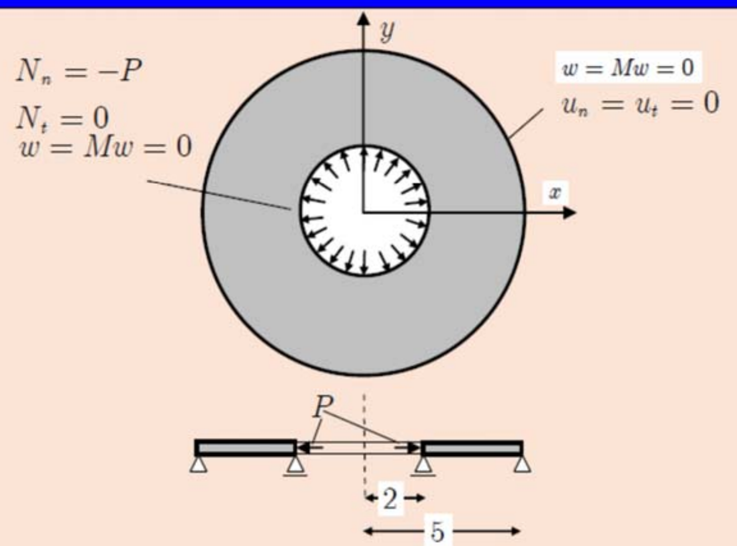
7.6Static and dynamic, Linear and nonlinear 2D viscoelastic problems

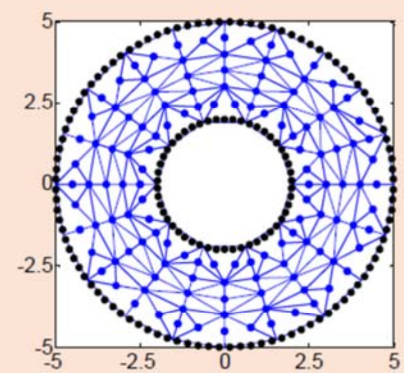
II. Some Solved Example Problems

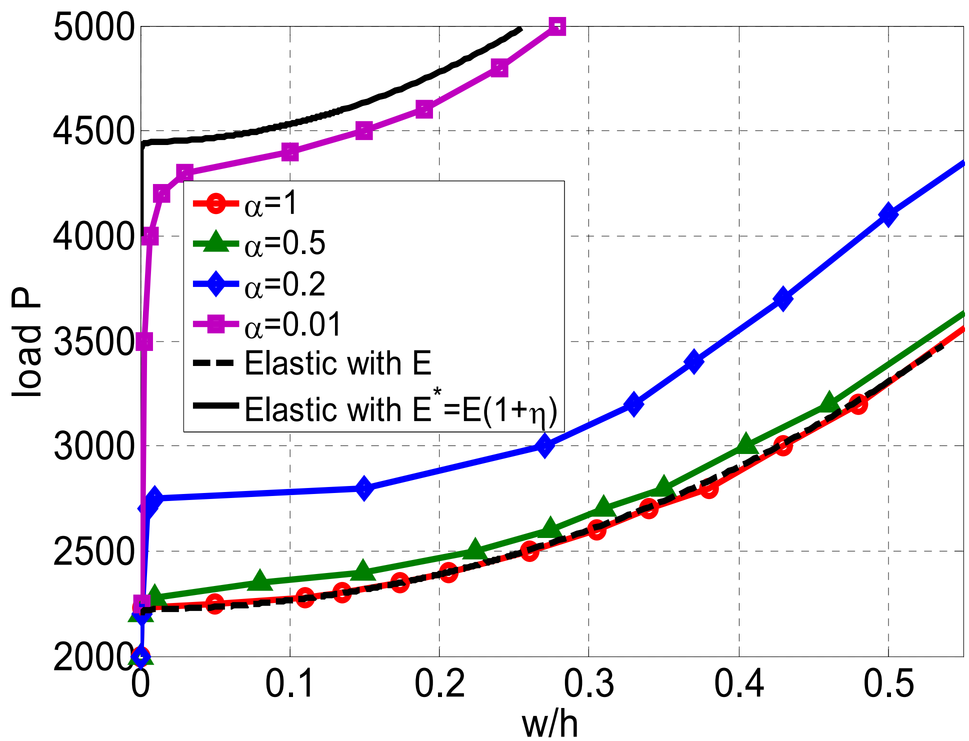
Example 1. The post-buckling response of the annular plate

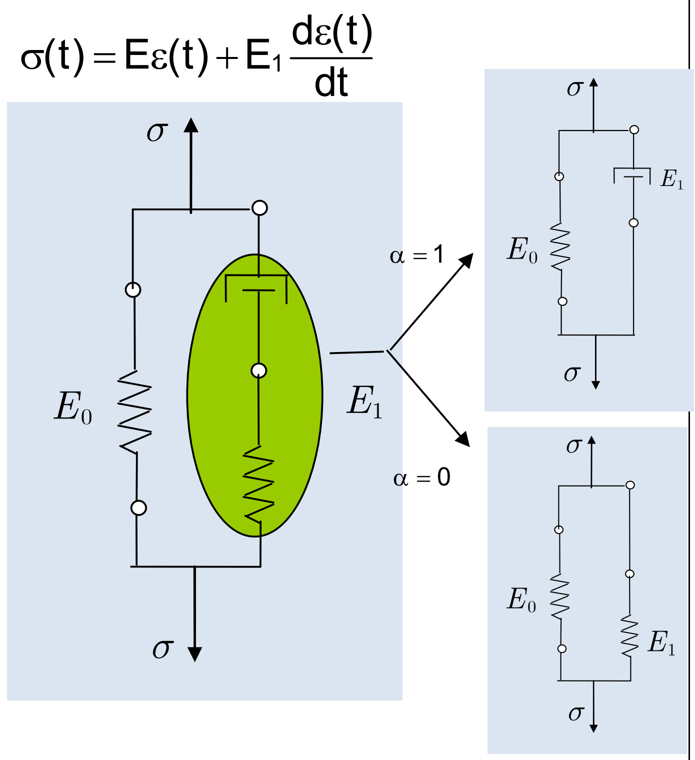
Katsikadelis Babouskos, EABE 2010

Geometry	Discretization









Load – deflection curves at large time at point $r = 3.5$ for various values of the order α of the fractional derivative and comparison with the elastic post-buckling paths.

Example 2. Resonance of a square viscoelastic plate.

Data	Geometry

BCs:

Simply supported $w = Mw = 0$,

(i) movable $N_x = N_y = N_{xy} = 0$

(ii) immovable $u = v = w = 0$

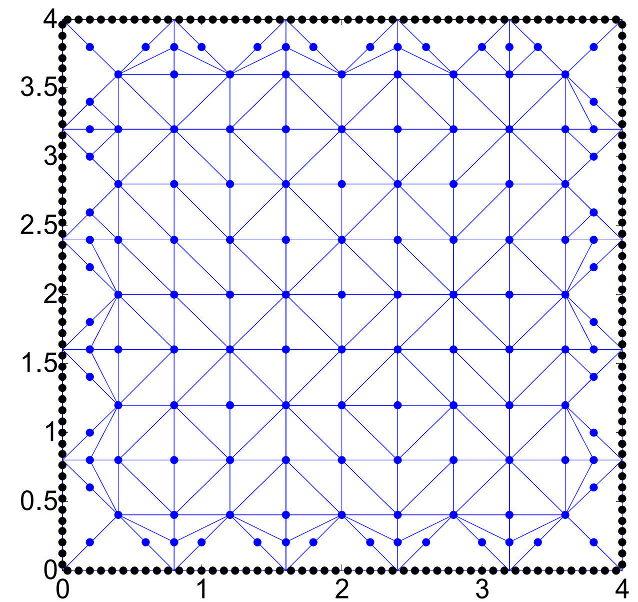
$E = 21 \times 10^6\ \text{N/m}^2, \rho = 10^4\ \text{kg/m}^3\quad \nu = 0.3$

Load $p_z = \cos(\Omega t)$.

$N = 204$ boundary elements and $M = 137$ internal nodes

Ritz method to reduce the degrees of freedom

40 Linear modes for Ritz reduction



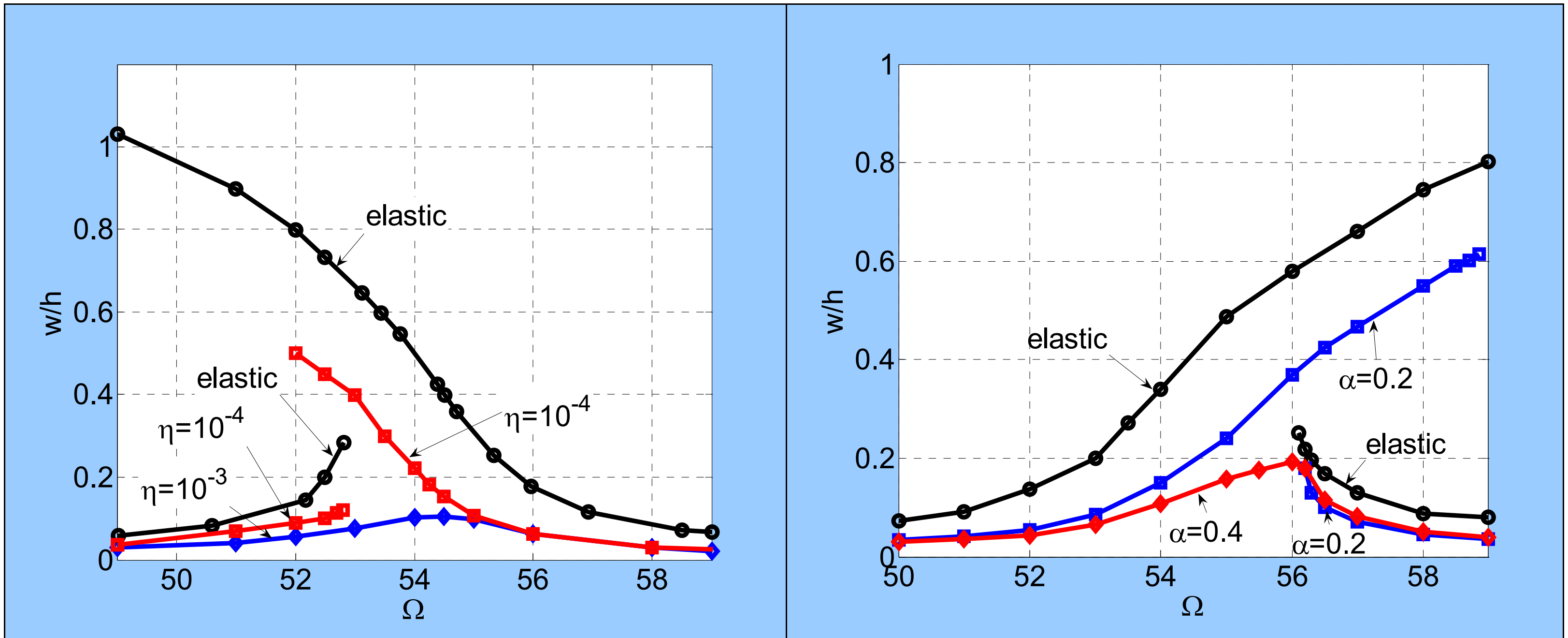


Figure 10: Amplitude-frequency curves for various values of the viscoelastic parameters and for the two cases of the inplane boundary conditions (i) movable (ii) immovable ($\eta = 1$).

The maximum deflection is taken at the steady state response. Excitation frequency $\Omega = 53.42$

The deflection increases and remains bounded due to the nonlinear character of the problem and the viscoelastic behavior of the material.

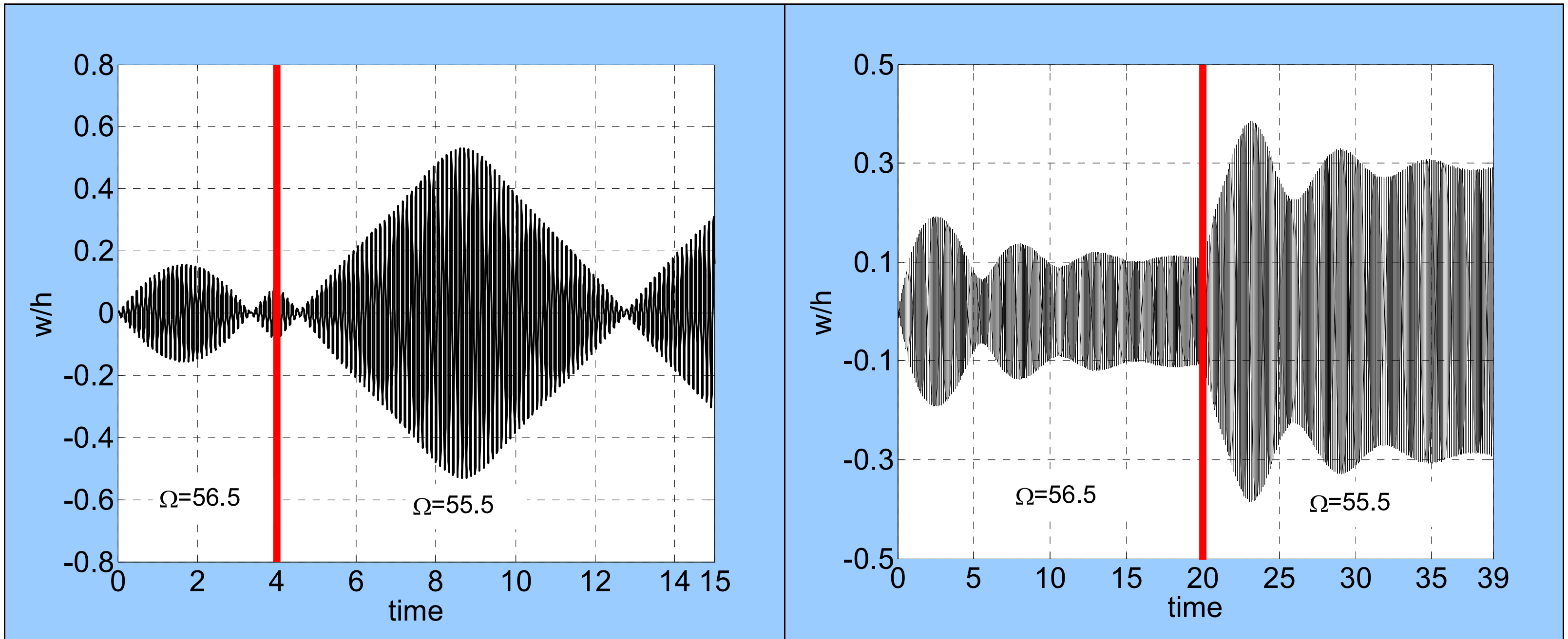


Figure 11: Time history of deflection at the center of the plate for two values of the excitation frequency ($\Omega_1 = 56.5$, $\Omega_2 = 55.5$): (a) elastic material, (b) viscoelastic material ($\alpha = 0.2$, $\eta = 0.01$, case (ii)).

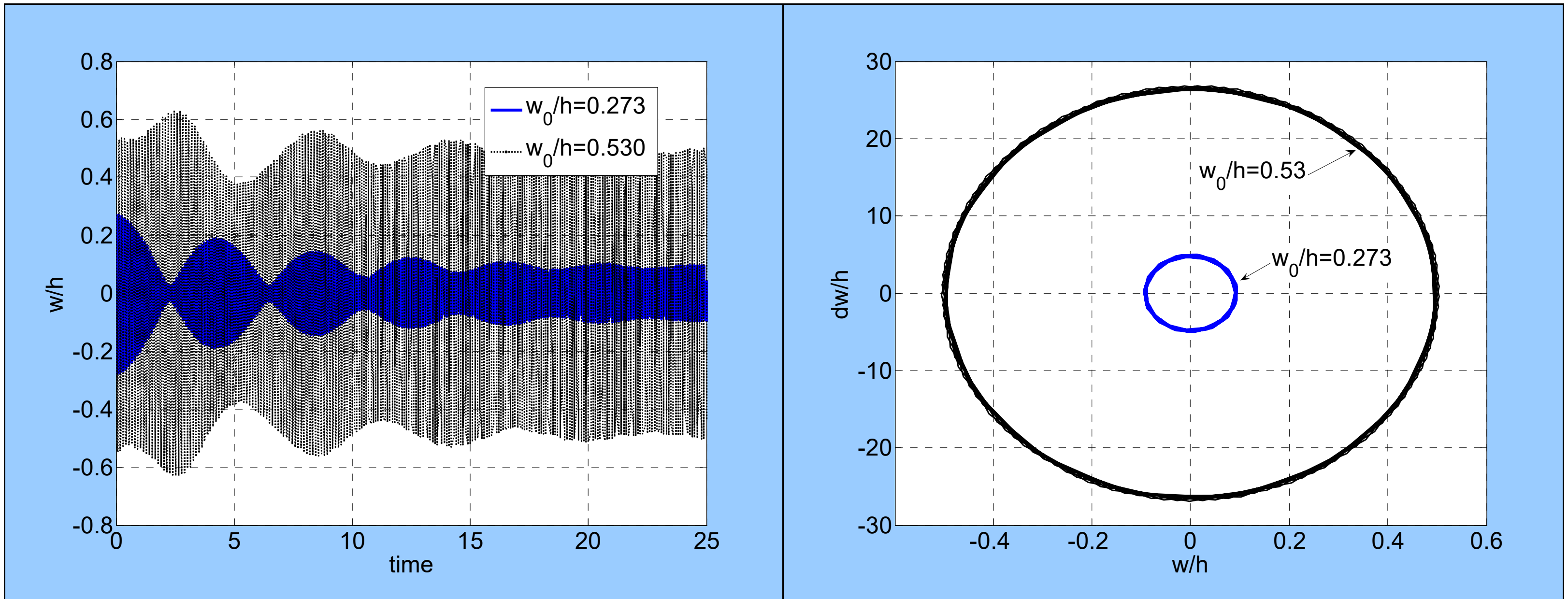
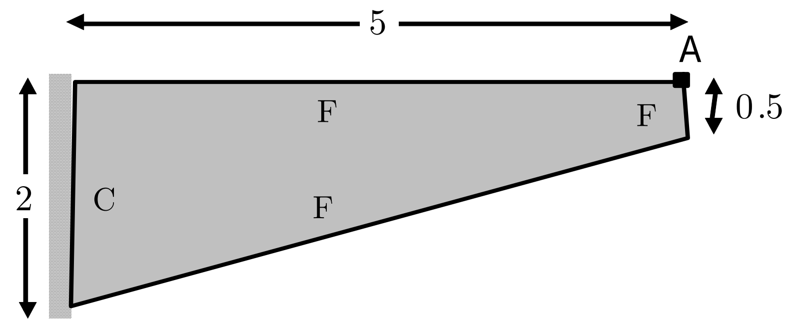
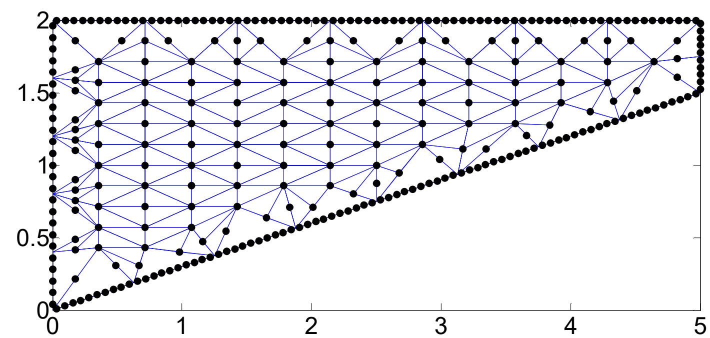
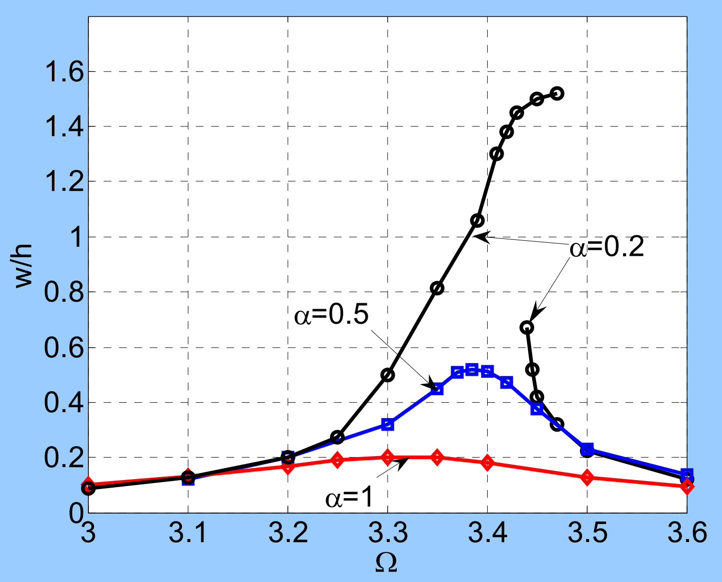


Figure 12: (a) Time history of the deflection at the center of the plate and (b) phase plane of the steady state response of the two stable solutions ($\alpha = 1, \eta = 0.001, \omega = 52$).

Example 3. Resonance of a wing form viscoelastic cantilever plate.

Data	Geometry
$h = 0.01\text{m}$ $E = 21\times10^6\ \text{N/m}^2, \rho = 7550\ \text{kg/m}^3,$ $\nu = 0.3$ Load $p_z = 0.0001k\cos(\Omega t)$. $N = 255$ boundary elements and $M = 131$ internal nodes Ritz method to reduce the degrees of freedom 40 Linear modes for Ritz reduction	 



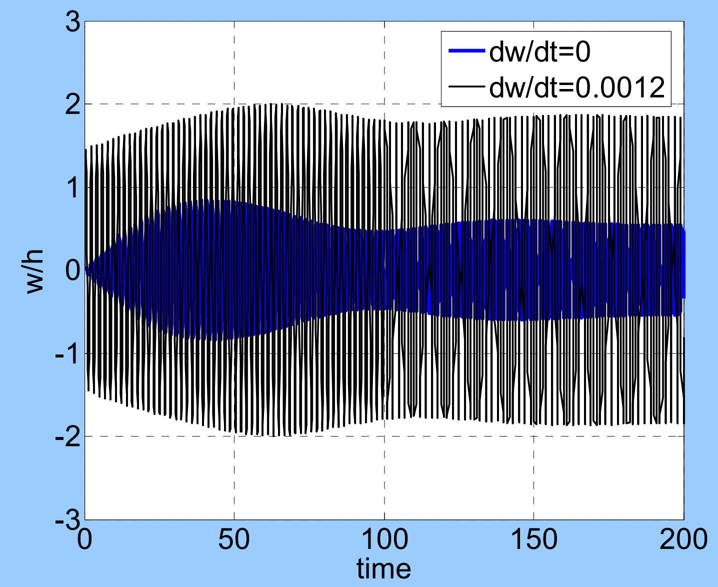


Figure 16: Amplitude-frequency curves for various values of the order α of the fractional derivative model ($k=1$).

Figure 18: Time history of the deflection at point A for different initial velocities ($\Omega = 3.45$, $k = 1.3$, $\alpha = 0.2$).

The IBVP for the plate in terms of the displacements

System of three coupled partial FDEs

$$C\{[u_{,x} + \nu v_{,y} + \frac{1}{2}(w_{,x}^2 + \nu w_{,y}^2) + \eta D_c^\alpha [u_{,x} + \nu v_{,y} + \frac{1}{2}(w_{,x}^2 + \nu w_{,y}^2)]\}_{,x}$$

$$+ C\frac{1-\nu}{2}\{(u_{,y} + v_{,x} + w_{,x} w_{,y}) + \eta D_c^\alpha (u_{,y} + v_{,x} + w_{,x} w_{,y})\}_{,y} - \rho h \ddot{u} = -p_x$$

$$C\frac{1-\nu}{2}\{(u_{,y} + v_{,x} + w_{,x} w_{,y}) + \eta D_c^\alpha (u_{,y} + v_{,x} + w_{,x} w_{,y})\}_{,x}$$

$$+ C\{[\nu u_{,x} + v_{,y} + \frac{1}{2}(\nu w_{,x}^2 + w_{,y}^2)] + \eta D_c^\alpha [\nu u_{,x} + v_{,y} + \frac{1}{2}(\nu w_{,x}^2 + w_{,y}^2)]\}_{,y} - \rho h \ddot{v} = -p_y$$

$$C\{[u_{,x} + \nu v_{,y} + \frac{1}{2}(w_{,x}^2 + \nu w_{,y}^2) + \eta D_c^\alpha [u_{,x} + \nu v_{,y} + \frac{1}{2}(w_{,x}^2 + \nu w_{,y}^2)]\}w_{,xx} + C(1-\nu)\{(u_{,y} + v_{,x} + w_{,x} w_{,y}) + \eta D_c^\alpha (u_{,y} + v_{,x} + w_{,x} w_{,y})\}w_{,xy}$$

$$+ C\{[\nu u_{,x} + v_{,y} + \frac{1}{2}(\nu w_{,x}^2 + w_{,y}^2)] + \eta D_c^\alpha [\nu u_{,x} + v_{,y} + \frac{1}{2}(\nu w_{,x}^2 + w_{,y}^2)]\}w_{,yy} - p_x w_{,x} - p_y w_{,y} + \rho h \ddot{u} w_{,x} + \rho h \ddot{v} w_{,y} - \rho h \ddot{w} = -p_z$$

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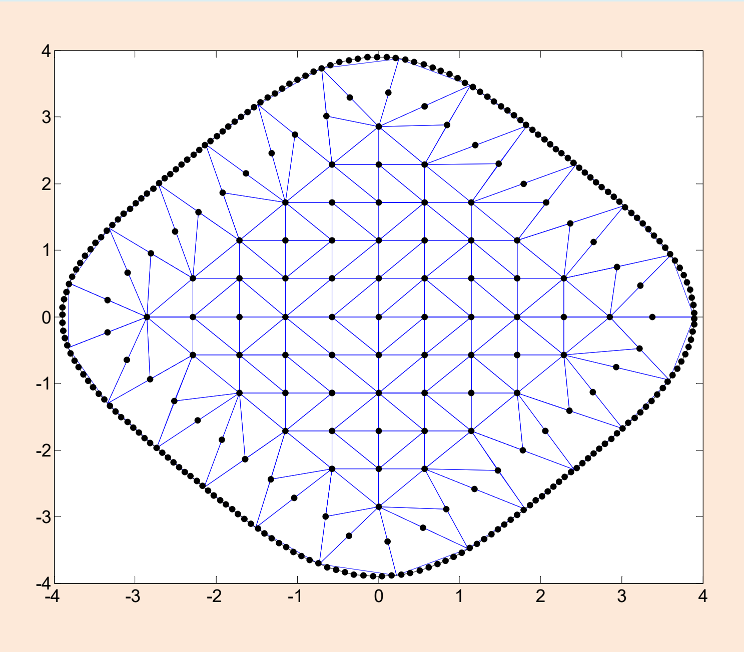
Example 4. Resonance of a viscoelastic membrane.

Data	Geometry and discretization
$a=3,\ b=1.3$ Thickness: $h=0.002\text{m}$, density $\rho/h=5000\text{kgr}/\text{m}^3$, Material Constants: $E=1.1\times10^5\,\text{kN}/\text{m}^2$, $\nu=0.3$. Prestress $u_n=0.2\text{m}$, $u_t=0\text{m}$ Load: $p_x=p_y=0,\quad p_z=p_0\sin(\Omega t)$	

$$r=(ab)^{1/2}\,/\left[(\cos\theta/a)^2+(\sin\theta/b)^2\right]^{(1/4)}\left[(\cos\theta/b)^2+(\sin\theta/a)^2\right]^{(1/4)},\quad 0\le\theta\le2\pi$$

Fractional Kelvin-Voigt model

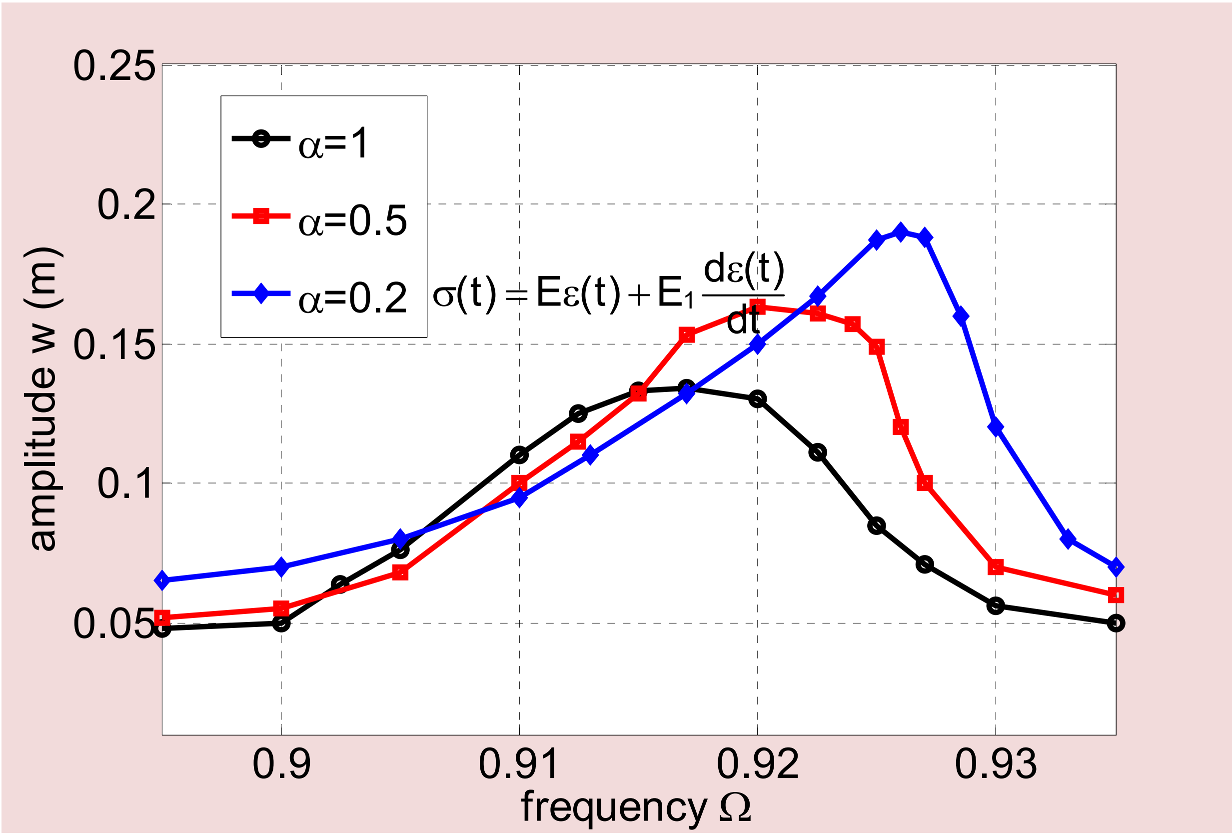


Figure 7. Amplitude-frequency curves for various values of the fractional derivative α $(\eta = 5)$

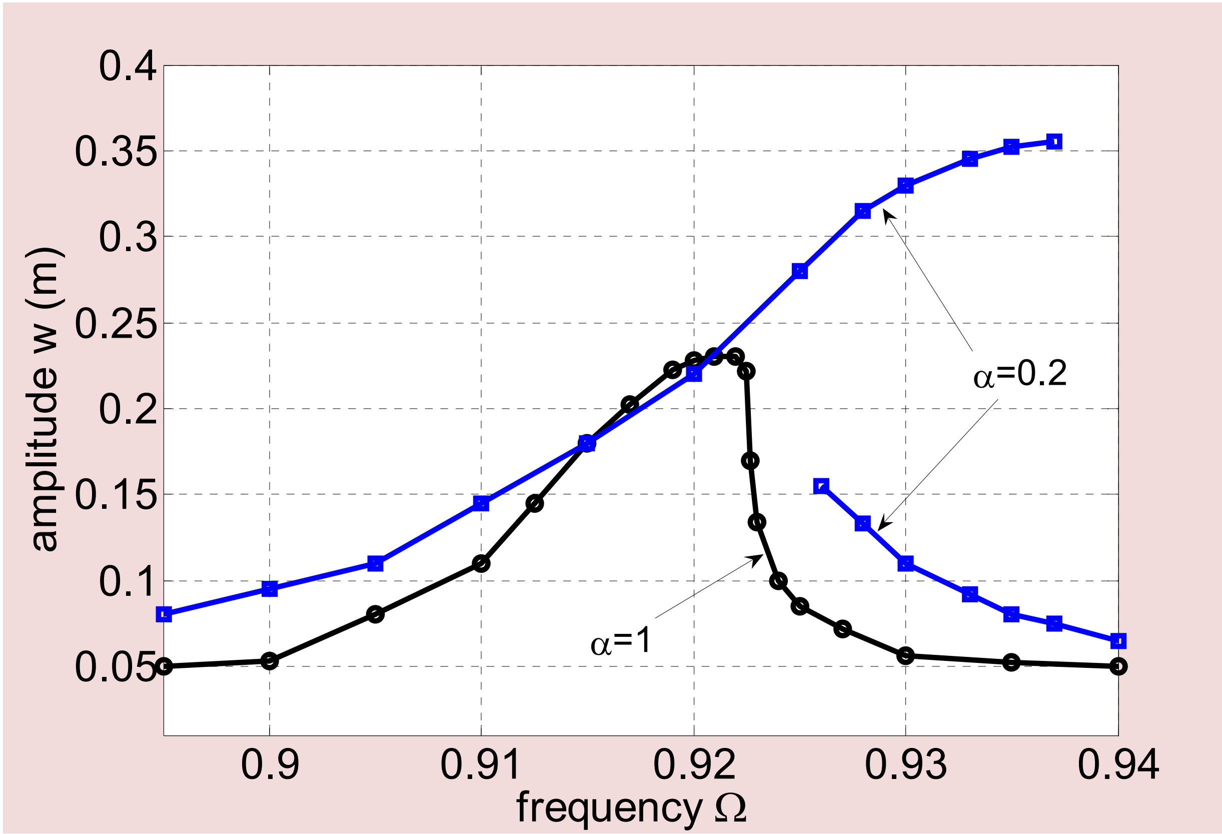


Figure 8. Amplitude-frequency curves for two values of the fractional derivative α $(\eta = 1)$

The fractional Diffusion-Wave Equation

The IBVP Governing Fractional PDE

Diff Equation

$$\rho D_c^{\beta} u + \eta D_c^{\alpha} u = Au_{,xx} + 2Bu_{,xy} + Cu_{,yy} + Du_{,x} + Eu_{,y} + Fu + g(\mathbf{x},t) \qquad \mathbf{x}(x,y) \in \Omega, \ t > 0$$

$$A = A(\mathbf{x}), B = B(\mathbf{x}), \ldots, F = F(\mathbf{x})$$

$0 < \alpha < \beta \le 2$ and $\rho = \rho(\mathbf{x})$, $\eta = \eta(\mathbf{x})$ and $g(\mathbf{x},t)$

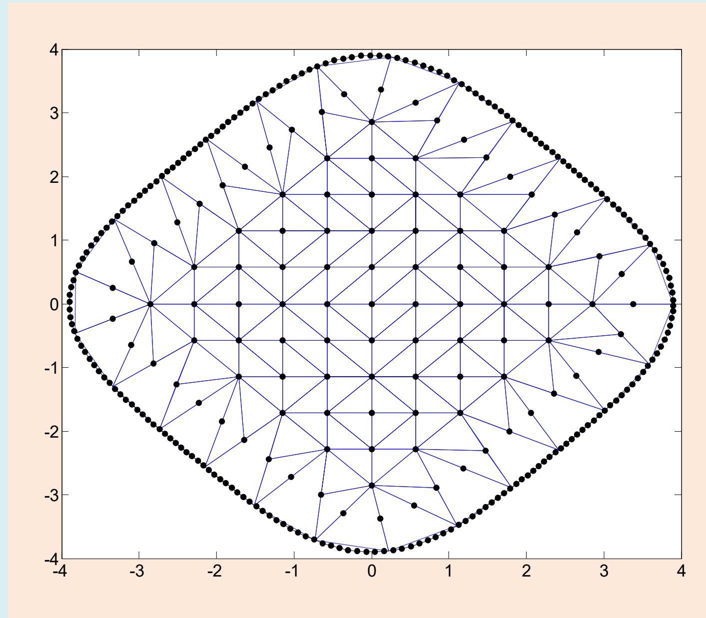
BCs on Γ

$$u = \alpha(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_a, \qquad k(\mathbf{x})u + \nabla u \cdot \mathbf{m} = \gamma(\mathbf{x},t), \ \mathbf{x} \in \Gamma_b$$

ICs in Ω

$u(\mathbf{x},0) = f_1(\mathbf{x})$ if $\beta \le 1$ or $u(\mathbf{x},0) = f_1(\mathbf{x})$, $\dot{u}(\mathbf{x},0) = f_2(\mathbf{x})$ if $\beta > 1$

Example 5. The Diffusion wave equation in plane body of arbitrary geometry.

Data	Geometry and discretization
$a=3,\ b=1.3,\ h=0.002,\ \beta=1.7,\ \alpha=0.8$ $A=(y^2-x^2+50)/50,$ $B=2xy/50,\ C=(x^2-y^2+50)/50,$ $D=E=0\ \ F=0;$ $\rho=5\exp(-0.1\{\|x\|+\|y\|\}),\ \eta=0.4(x^2+y^2)^{1/2}$ $g=\rho D_c^{1.7}T+\eta D_c^{0.8}T-(AU_{xx}+2BU_{xy}+CU_{yy})$ $T(t)=t-t^3/6+t^5/200$ $U(x,y)=a^2b^2-\left[(x/a)^2+(y/b)^2\right]\left[(x/b)^2+(y/a)^2\right]$ Bcs: $u(\mathbf{x},t)=0,\ \mathbf{x}\in\Gamma$, ICs: $u(\mathbf{x},0)=0,\ \dot{u}(\mathbf{x},0)=U(x,y)$ exact solution $u_{exact}=T(t)U(x,y)$.	

$$r=(ab)^{1/2}/\left[(\cos\theta/a)^2+(\sin\theta/b)^2\right]^{(1/4)}\left[(\cos\theta/b)^2+(\sin\theta/a)^2\right]^{(1/4)},\quad 0\le\theta\le 2\pi$$

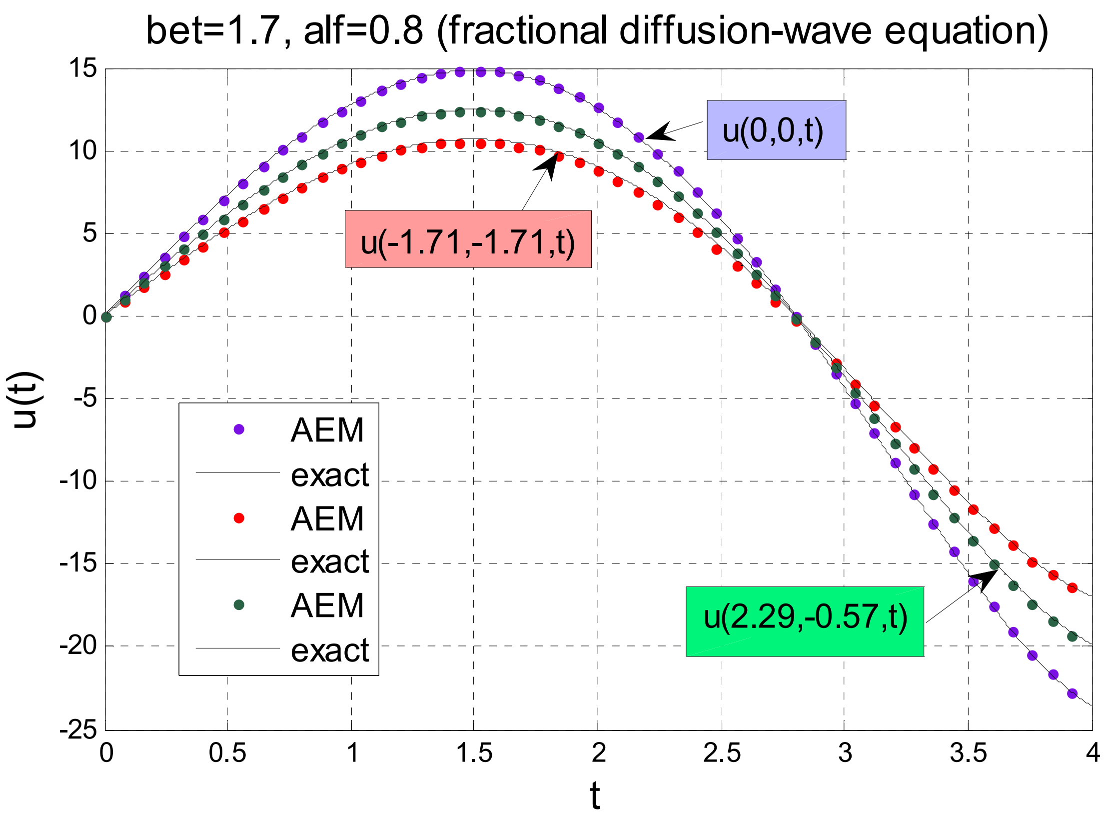


Fig. 11. Fractional Diffusion-Wave Equation.

Time history at three points

II Fractional Differential Equations
b. Distributed Order

The distributed-order fractional derivative

$$\int_a^b F[p, D_c^p u(t)]dp$$

General nonlinear distributed-order fractional differential equation

$$\int_a^b F[p, D_c^p u(t)]dp + G[t, u(t), D_c^{\alpha_i} u(t)] = f(t)$$

$F, G :=$ nonlinear functions

Linear distributed-order fractional differential equation

$$\int_{0.2}^{1.5} \Gamma(3-p) D_c^p u(t)dp = 2\frac{t^{1.8} - t^{0.5}}{\ln t}, \qquad u(0) = \dot{u}(0) = 0$$

Solution of distributed-order fractional differential equation

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Numerical solution of distributed order fractional differential equations

John T. Katsikadelis

The fundamental idea is to replace the integral by a sum and treat the resulting equation a multi-term FDE, e.g.

$$\int_{\alpha}^{\beta} \phi(p) D_c^p u(t)\,dp = f(t), \quad 0 \le \alpha < \beta \le 2 \tag{3}$$

The initial conditions depend on ceil(β). Thus we have

$$u(0) = u_0 \quad \text{if } 0 < \beta \le 1 \tag{4a}$$

or

$$u(0) = u_0,\ \dot{u}(0) = \dot{u}_0 \ \text{ if } 1 < \beta \le 2 \tag{4b}$$

2.1 Integration interval $[0,\beta]$

Approximating the integral in Eq. (3) with a sum using the trapezoidal rule with K equal intervals we obtain

$$\Delta p\left[\frac{\phi_0}{2}D_c^{p_0}u+\phi_1 D_c^{p_1}u+\phi_2 D_c^{p_2}u+\cdots+\phi_{K-1}D_c^{p_{K-1}}u+\frac{\phi_K}{2}D_c^{p_K}u\right]=f(t),\ \Delta p=\beta/K \qquad (5)$$

with $D_c^{p_0}u=u$ and $D_c^{p_K}u=D_c^{\beta}u$.

Example: The fractional distributed-order (FDO) oscillator (Katsikadelis, 2014)

$$u^{(2)}(t)+\sigma(t)+\omega^2 u(t)=g(t) \qquad (37a)$$

$$\int_0^1 \phi(p)D_c^p\sigma dp=\lambda\int_0^1 \psi(p)D_c^p u dp \qquad (37b)$$

$$u(0)=\dot{u}(0)=0 \qquad (37c)$$

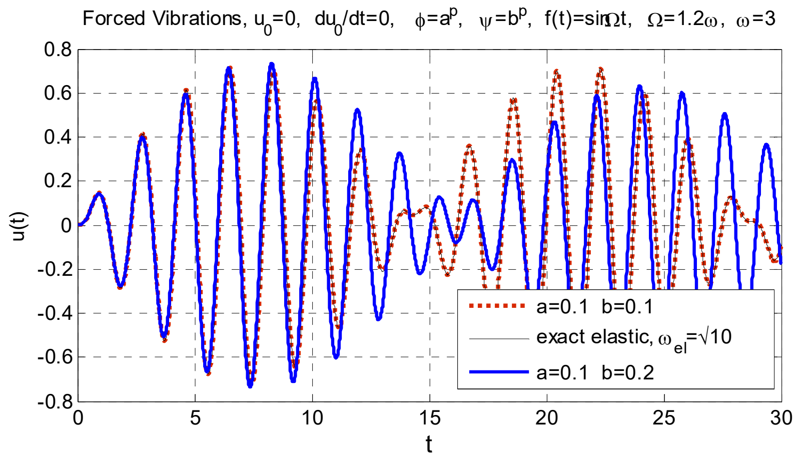


Fig. 8 Response of the FDO oscillator in Example 4.4.

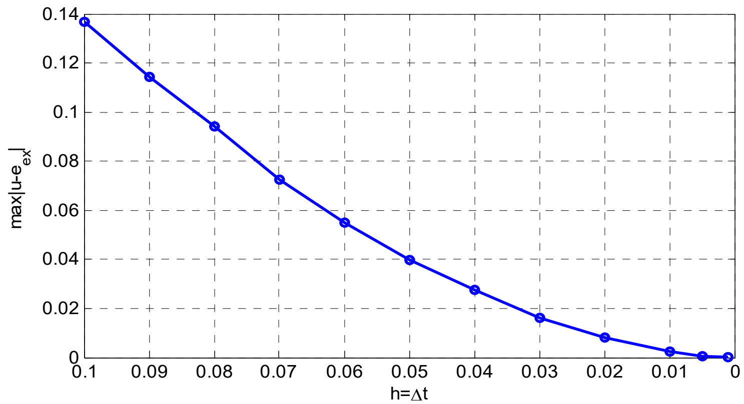


Fig. 9. Maximum error versus integration time step $h = \Delta t$ in Example 4.4.

II Fractional Differential Equations
c. Variable Order

The fractional calculus has allowed the definition of any order fractional derivative (FD), real or imaginary. This fact enables us to consider the fractional derivative to be an explicit function of time (explicit VO-FD) or of some other dependent variable (implicit VO-FD).

Although the extension from constant-order to VO-FD may seem somewhat natural and several systems have been modeled with VO-FD, this idea has been forwarded only very slowly [12]. In a sense, this extension may be have been prevented by the difficulty to obtain solutions to VO-FDEs.

Without excluding other types of VO FDs, we adopt the Caputo type VO order FD of a function $u(t)$

$$D^{\alpha(t)}u(t)=\frac{1}{\Gamma[1-\alpha(t)]}\int_0^t (t-x)^{-\alpha(t)}\dot{u}dx\,, \qquad 0<\alpha(t)<1,\ t>0 \tag{2.1}$$

$$\lim_{a\to 1} D^{\alpha(t)}u(t)=\dot{u}(t)\,, \qquad \lim_{a\to 0} D^{\alpha(t)}u(t)=u(t)-u_0 \tag{2.2}$$

Explicit: $\alpha = \alpha(t)$ depends only of time

Implicit: $\alpha = \alpha[u(t)]$a or $\alpha = \alpha[\dot{u}(t)]$ depends on the field function and/or its derivative

Apparently, the study of the response of systems modelled with VO FDs requires the solution of VO-order fractional differential equations. An efficient method is described in: Katsikadelis J.T. "Numerical solution of variable order fractional differential equations", arXiv:1802.00519v2 [math.NA]. This method is presented concisely

Example 1

Compute the VO-FD of the function $u(t) = t^2$, $t \in [0,1]$,

(i) $\alpha = (50t + 49)/100$, (ii) $\alpha = 1 - \exp(-t)$.

(i) The exact VO-FD is

$$D^{\alpha(t)}(t^2) = \frac{1}{\Gamma[1-\alpha(t)]}\int_0^t (t-x)^{-\alpha(t)}\dot{u}dx = \frac{1}{\Gamma[1-\alpha(t)]}\frac{20000t^{\frac{151}{100}-\frac{t}{2}}}{(50t-151)(50t-51)} \quad (1)$$

Fig. 2 shows the exact versus the approximate VO FD as computed using Eq. (2.4) and the error for $h = 0.001$. ($\max(|\,\text{error}|) = 9.21514\text{e-}4$)

(ii) The exact VO-FD is

$$D^{\alpha(t)}(t^2) = \frac{1}{\Gamma[1-\alpha(t)]}\int_0^t (t-x)^{-\alpha(t)}\dot{u}dx = \frac{1}{\Gamma[1-\alpha(t)]}\frac{2\exp(2t)t^{\exp(-t)+1}}{\exp(t)+1} \quad (2)$$

Fig. 3 shows the exact versus the approximate VO-FD and the error for $h = 0.001$,. $\max(|\,\text{error}|) = 4.62050\text{e-}05$

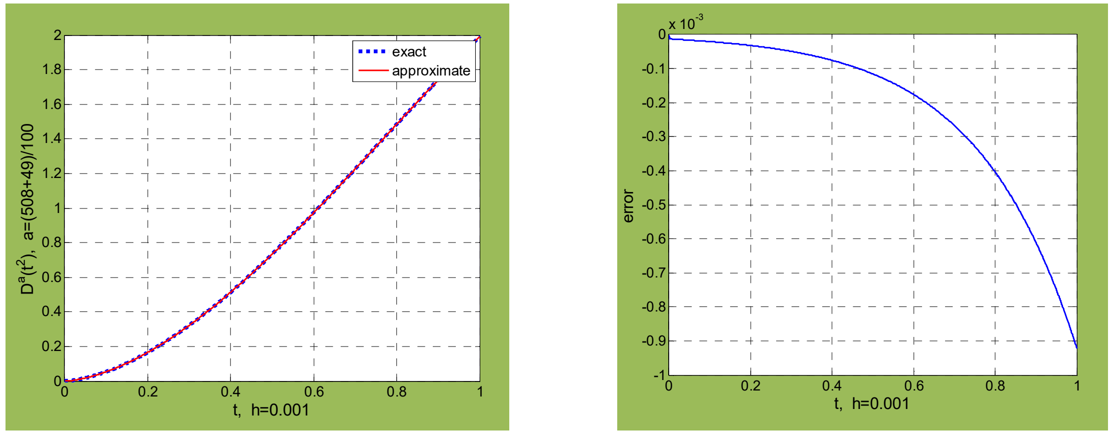


Figure 2. VO-FD in Example 1 (i). $D^{\alpha(t)}(t^2)$, $\alpha=(50t+49)/100$

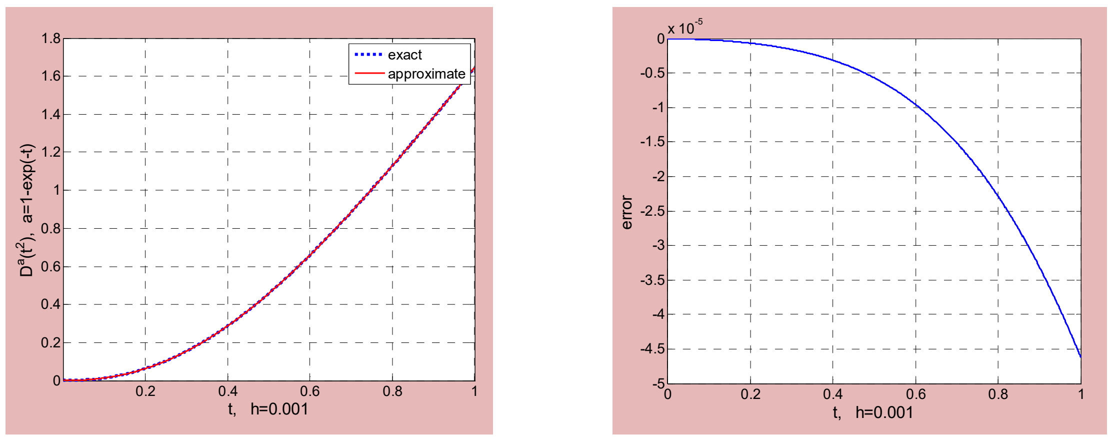


Figure 3. VO-FD in Example 1 (ii). $D^{\alpha(t)}(t^2)$, $\alpha=1-\exp(-t)$

3 Linear VO-FD Equations

3.1. Explicit VO-DF equations

The solution procedure is illustrated by solving the second order VO-FDE describing the response of the oscillator with VO fractional damping

Example 2. Explicit VO-DF equation

Solve the initial value problem

$$a_1\ddot{u} + a_2 D^{a(t)}u + a_3 u = p(t), \quad u_0 = 1, \dot{u}_0 = 10, \quad a(t) = d - k\exp(-t) \qquad (1)$$

Assume: $a_1 = 1, a_2 = 2\xi\omega, a_3 = \omega^2$, $\xi = 0.1$, $\omega = 5$, $p(t) = 0$ and d, k as follows:

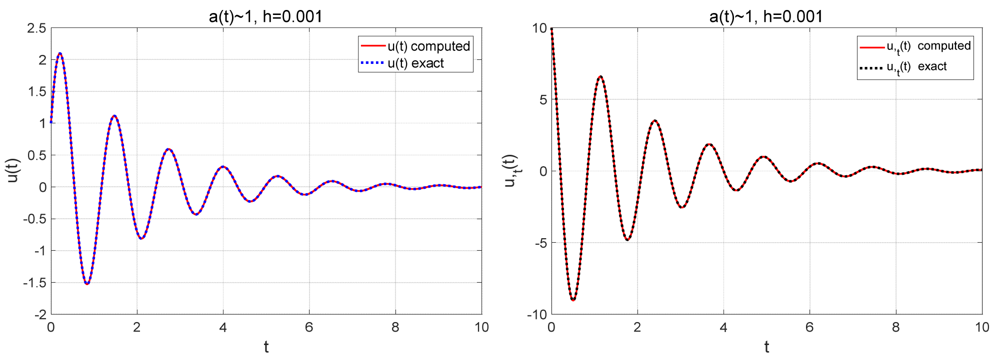
(i) $d = 0.9999, k = 10e-10$. In this case it is $a(t) \approx 1$, hence as anticipated $D^{a(t)}u \approx \dot{u}$, and the computed solution approximates the exact solution

$$u_{ex} = \exp(-\xi\omega t)(\frac{\dot{u}_0 + \xi u_0}{\omega_D}\sin\omega_D t + u_0\cos\omega_D t), \quad \omega_D = \omega\sqrt{1-\xi^2} \qquad (2)$$

(ii) $d = 1e-10, k = 1e-10$. In this case, it is $a(t) \approx 0$, hence as anticipated $D^{a(t)}u \approx u - u_0$, and the computed solution approximates the exact solution

$$u_{ex} = \frac{\dot{u}_0}{\omega}\sin\omega t + u_0\cos\omega t + a_2\frac{u_0}{k}, \qquad k = a_2 + a_3, \qquad \omega = \sqrt{k / a_1} \tag{3}$$

(iii) Solve Eq. (1) for: $a = 1$; $a = 1 - \exp(-t)$; $a = 0.8$; $a = 0.8[1 - \exp(-t)]$; $a = 0.5[1 - \exp(-t)]$. The computed results are plotted in Fig. 6. As anticipated, the VO FD yields larger displacements than the corresponding constant order FD.



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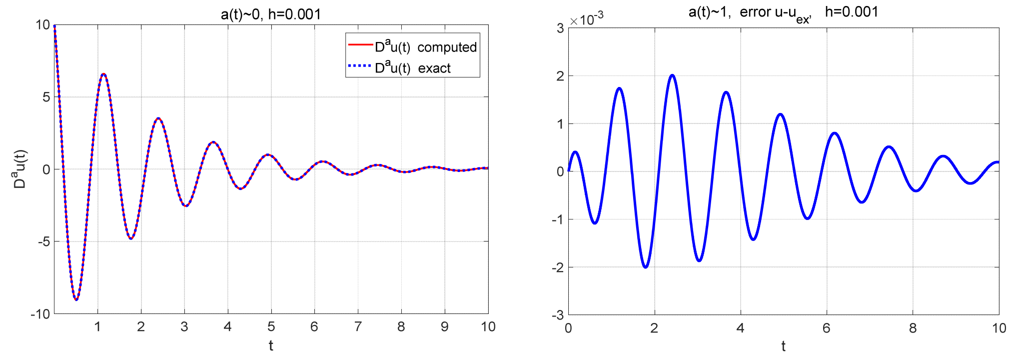
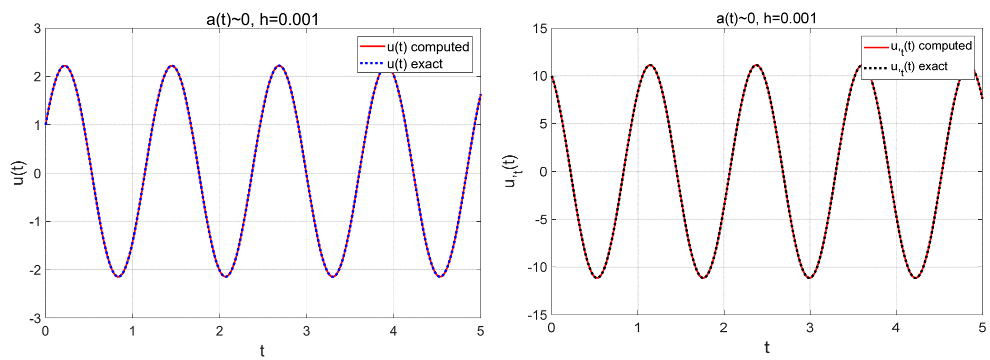


Figure 4. Results in Example 2: case (i)



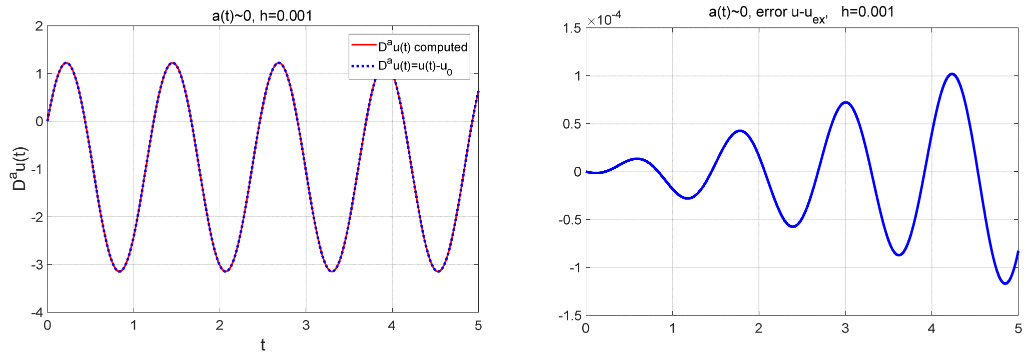


Fig. 5 Results in Example 2: case (ii)

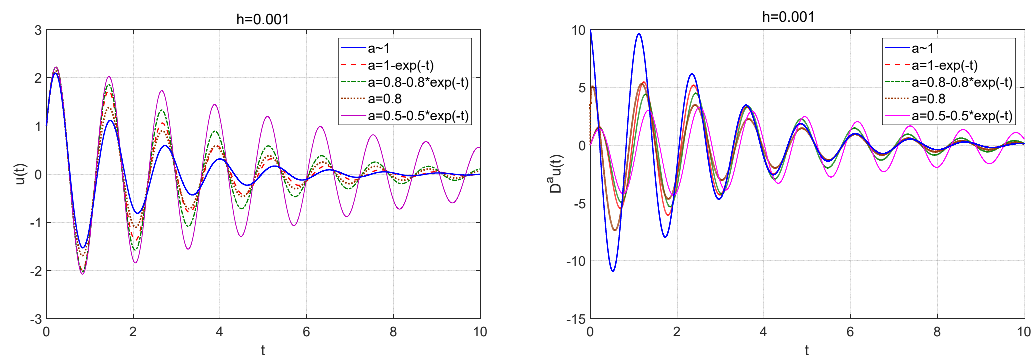


Fig. 6 Results in Example 2: case (iii)

In all studied cases the stability condition, Eq. (3.11), was satisfied. Fig. 7 shows the spectral radius for $a = 0.8[1 - \exp(-t)]$.

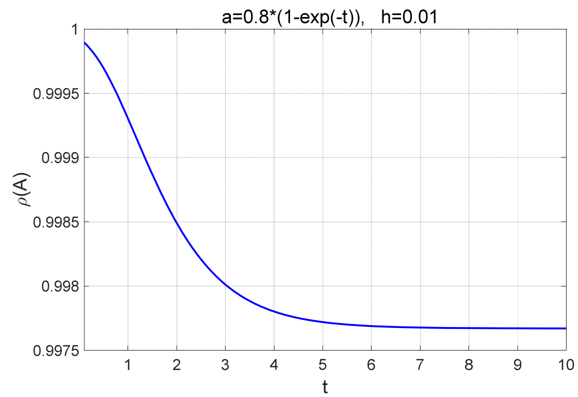


Fig. 7. Spectral radius in Example 2, $a = 0.8[1 - \exp(-t)]$.

3.2. Implicit VO-DF equations

The solution procedure is illustrated by solving the second order VO-FDE describing the response of the oscillator with VO fractional damping depending on the velocity

Example 3

Solve the initial value problem

$$a_1\ddot{u} + a_2 D^{a(t)}u + a_3 u = p(t), \qquad a(t) = d - k\tanh(|\dot{u}|) \tag{1}$$

Assume: $a_1 = 1, a_2 = 2\xi\omega, a_3 = \omega^2$, $\xi = 0.1$, $\omega = 2$, $p(t) = 0$, and $u_0, \dot{u}_0$, d, k as follows:

(i) $d = 0.9999, k = 10e-10$ $u_0 = 0, \dot{u}_0 = 1$. In this case it is $a(t) \approx 1$, hence as anticipated $D^{a(t)}u \approx \dot{u}$ and the computed solution approximates the exact solution (2) in Example 2. This is shown in Fig 8.

(ii) $d = 1e-10, k = 1e-10$, $u_0 = 0, \dot{u}_0 = 1$. In this case, it is $a(t) \approx 0$, and the computed solution approximates the exact solution (3) in Example 2. This is shown in Fig 9.

(iii) When $u_0 = 0, \dot{u}_0 = 10$, compute the response for: $a(t) = 1; a = 1 - 0.5\tanh(|\dot{u}|)]$. The computed results are plotted in Fig. 10.

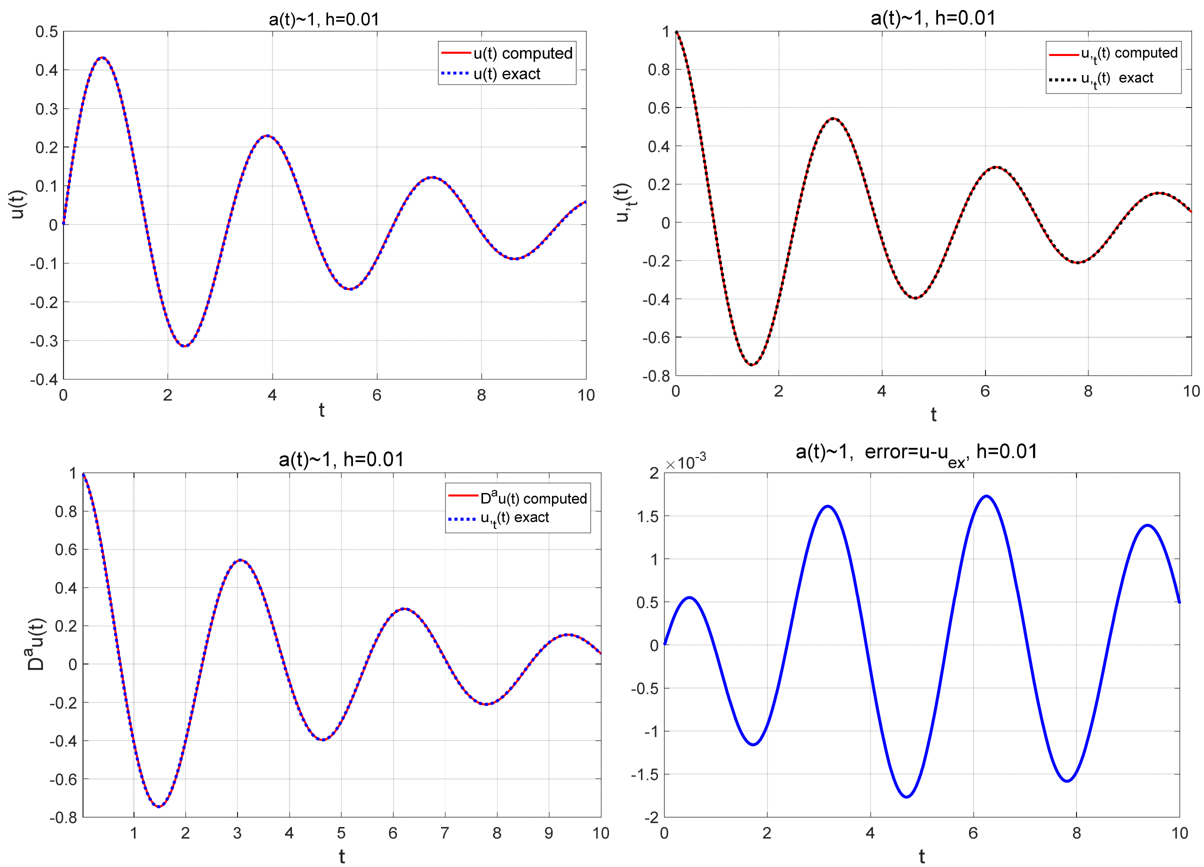


Fig. 8 Results in Example 3: case (i)

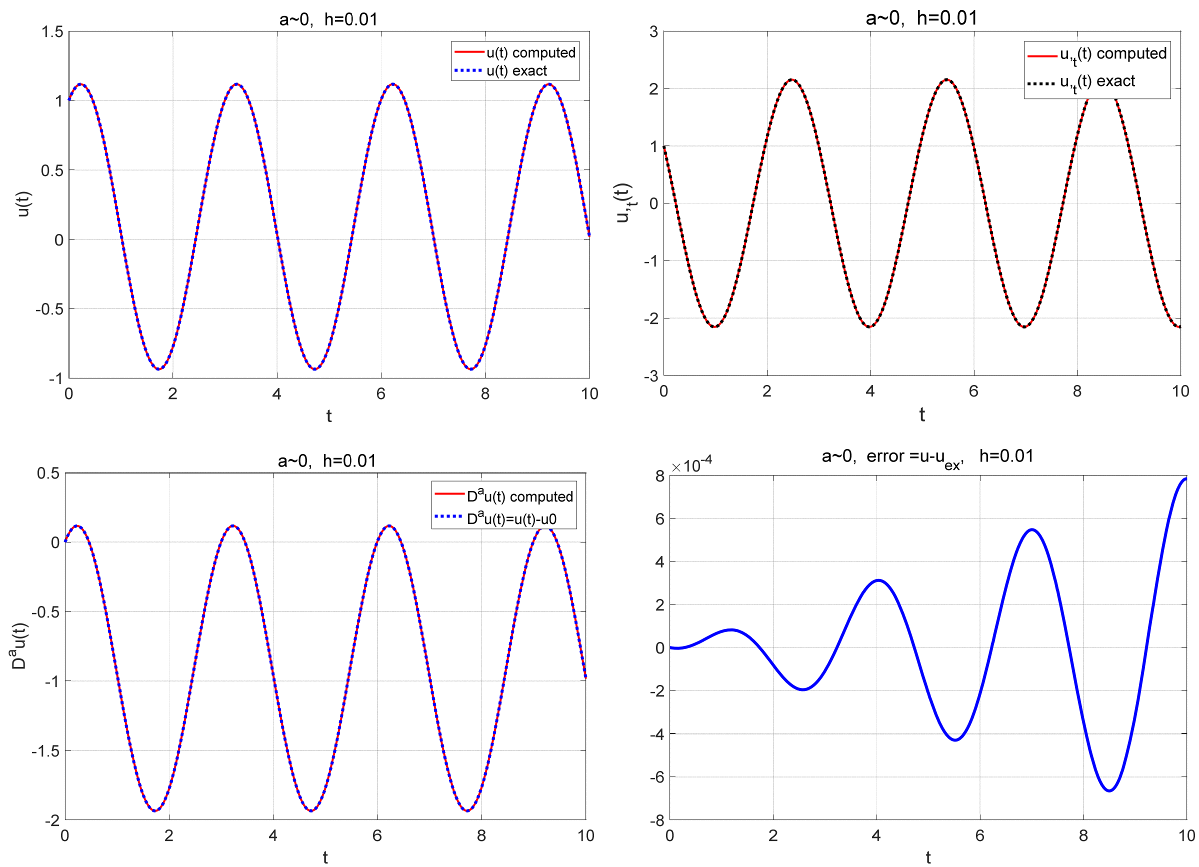


Fig. 9 Results in Example 3: case (ii)

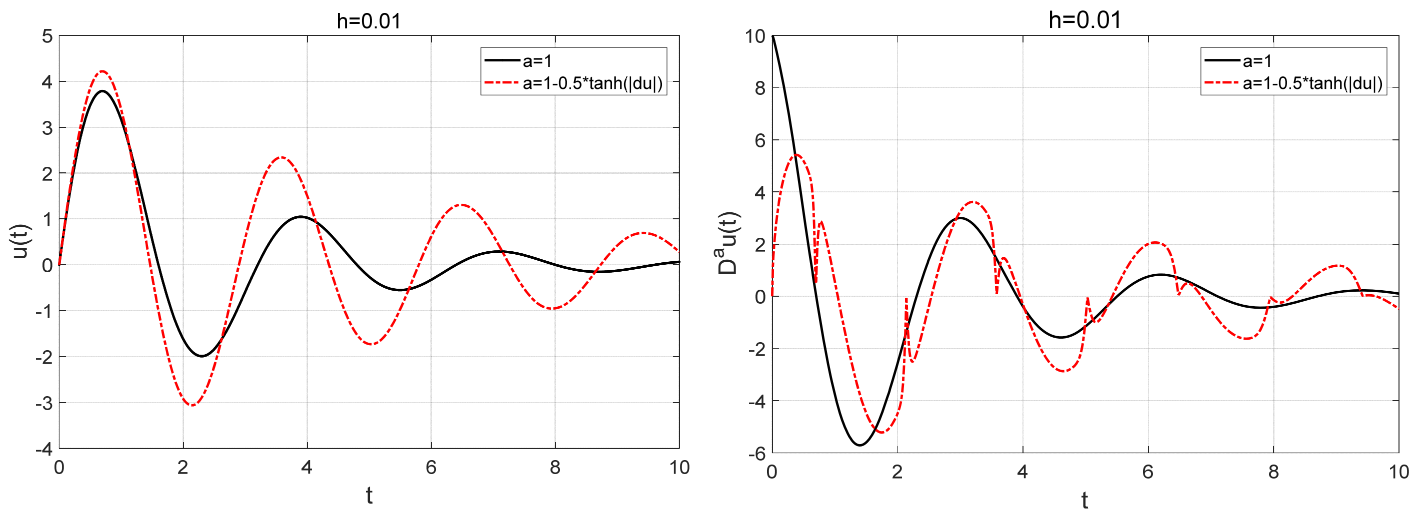


Fig. 10 Results in Example 3: case (iii)

4 Nonlinear VO-FD Equations

The solution is obtained using the same algorithm as in linear implicit VO FDEs.

Example 4

The numerical scheme is employed to solve the initial value problem for the fractional Duffing oscillator

$$\ddot{u} + 0.2D^{a(t)}u + u + u^3 = p(t), \quad a = 1 - \exp(-t) \tag{1}$$

$$u_0 = 0, \quad \dot{u}_0 = 0 \tag{2}$$

For $p(t) = 2 + t^2 + t^6 + 0.2\frac{1}{\Gamma(1-a)}\frac{2\exp((2t)t^{\exp(-t)+1}}{\exp(t)+1}$, Eq (1) admits an exact solution $u_{ex}(t) = t^2$.

The computed results are plotted in Fig. 11.

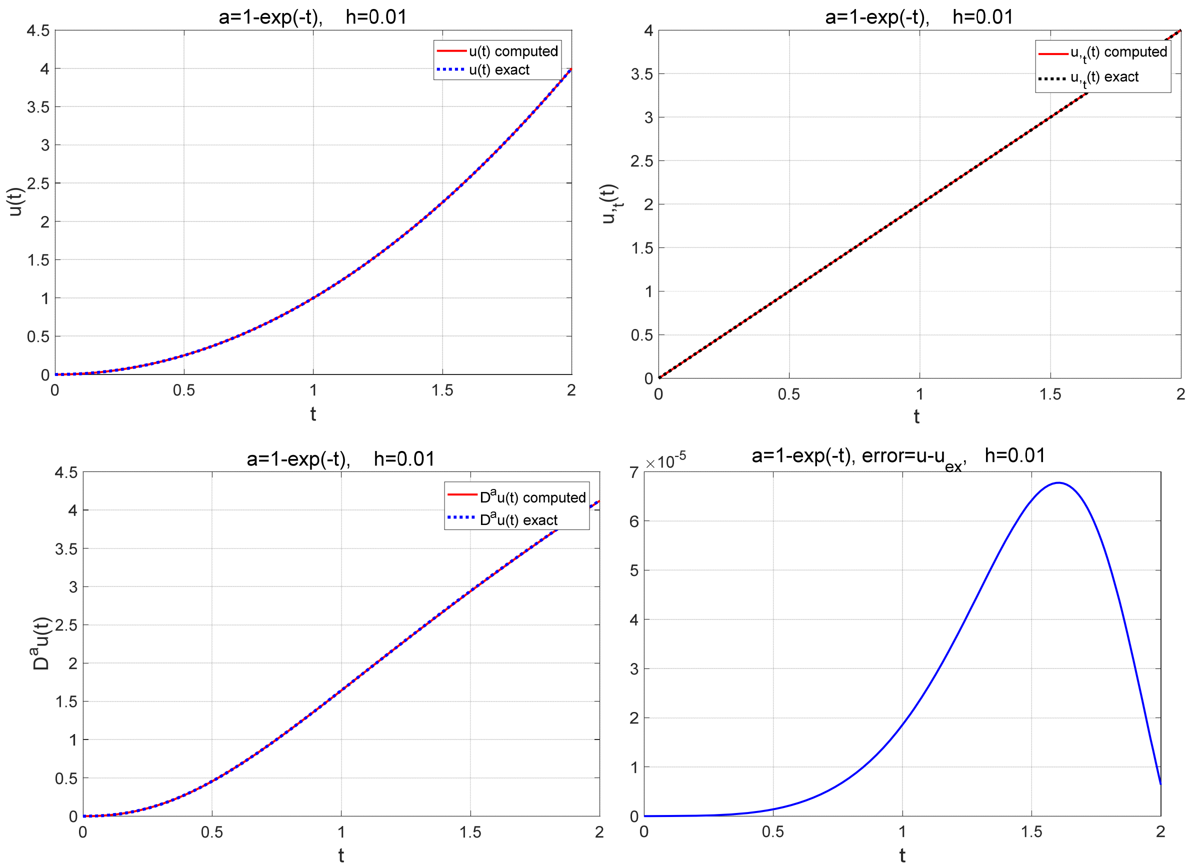


Figure 11. Results in Example 4.

5 VO-FD Equations with variable coefficients

So far we have developed the method for the solution of VO FDEs with constant coefficients. Obviously, if the coefficients a_1, a_2, a_3 are functions of the independent variable t, the previously described solution procedures remain the same except that the coefficients a_1, a_2, a_3 are evaluated in each step. In the following, the effectiveness of the method is demonstrated by solving the initial value problem in the example bellow

Example 5

Solve the VO FDE

$$(1+t^2)2+0.1t^{1/2}D^a u+(10+e^{-t})u=p(t), \qquad a(t)=1-0.5\exp(-t) \tag{1}$$

$$u_0=1,\ \dot{u}_0=1 \tag{2}$$

For $p(t)=[(1+t^2)+0.01t^{1/2}(\Gamma(1-a,0)-\Gamma(1-a,t))/\Gamma(1-a))+(10+e^{-t})]e^t$, Eq (1) admits an exact solution $u_{ex}(t)=e^t$. The computed solution is plotted in Fig. 12 as compared with the exact one.

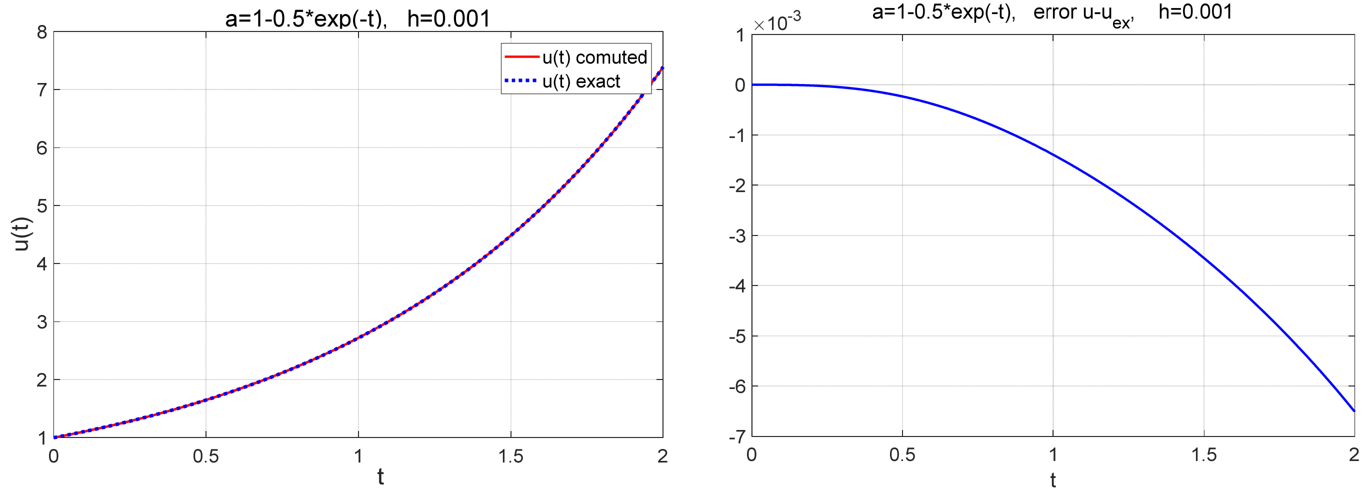


Figure 12. Results in Example 5.

IV. Conclusions

- The fractional derivatives enable the representation of physical systems with models approaching their actual response.
- Several fractional derivatives have been proposed. However, those permitting the application of physical initial and boundary conditions are preferable.
- Integer order derivatives have in some sense prevented the development of science.
- Variable order fractional derivatives provide a promising means of describing real world. It seems that we can circumvent phenomena resulting from the use of integer and constant order fractional Calculus.
- Simple and Robust Numerical methods for the solution of VO FDEs have developed for treating realistic physical problems.
- It would be interesting to develop a new description of the geometry based on the fractional derivative and investigate the response of the Universe via fractional formulation of the equations of general relativity.

Thank you for your attention